Saturated control of infinite-dimensional systems

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International Workshop on Robust LPV Control Techniques
and Anti-Windup Design
Toulouse, April 2018
Page 63: Natural frequency with "good and bad vibrations"

David A. Lind et Scott P. Sanders, The Physics of Skiing: Skiing at the Triple Point, 2nd edition; 2013
One way to kill bad vibrations?

Control your skis with smart materials!

Use passively piezoelectric patches
[L. Rothemann, H. Schretter, Active vibration damping of the alpine ski; 2010]

How to do it actively?
Need to consider a distributed parameter systems:
How to control the flexible ski structure? Euler Bernoulli equation:

\[ \rho \frac{\partial^2 w}{\partial t^2} + YI \frac{\partial^4 w}{\partial x^4} = \text{piezo force under control} \]
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Another domain with large flexible structures:
- satellites with large flexible structures
- and large airplanes with flexible wings and fluid dynamics

Flexible structure+ sloshing modes

can be controlled of distributed parameters systems (PDE)
with
- robustness
- experiments
- in-domain control

See [Robu, Baudouin, CP; 12], [Robu, Baudouin, CP, Arzelier; 12]

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See [Robu, Baudouin, CP; 12], [Robu, Baudouin, CP, Arzelier; 12]
Can we use saturated control?
Given a PDE, there exists now a large variety on methods to design linear controllers. It is well known that saturation can reduce the performance or even destabilize the system, even for finite-dimensional systems.

More precisely, even if
\[ \dot{z} = Az + BKz \]  
(1)
is asymp. stable, it may hold that
\[ \dot{z} = Az + \text{sat}(BKz) \]  
(2)
is not globally asymptotically stable.

It may exist new equilibrium, new limit cycles...

See e.g. [Tarbouriech, Garcia, Gomes da Silva Jr., Queinnec; 2011]

Goal of this talk:
What happens if in (2), instead of matrices $A$, $B$..., we have operators? More precisely, what happens if $A$ generates a semigroup and $B$ is a bounded control operator? An example of such a nonlinear PDE given by (2):
Wave equation with saturating in-domain control
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Wave equation with saturating in-domain control
Two objectives
  
  - Well-posedness
  - Stability

of the wave equation in presence of a disturbed saturating control with a Lyapunov method.
1. Well-posedness and stability of linear wave equation with a saturated in-domain control
   
   Lyapunov method, LaSalle invariance principle

2. Well-posedness and stability of linear abstract systems with a saturated in-domain control

   strict Lyapunov method, robustness result

3. Numerical simulations on wave equation

   effect of the saturation level

4. Conclusion
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Outline

1. Well-posedness and stability of linear wave equation with a saturated in-domain control
   Lyapunov method, LaSalle invariance principle

2. Well-posedness and stability of linear abstract systems with a saturated in-domain control
   strict Lyapunov method, robustness result

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1D wave equation with in-domain control.

Dynamics of the vibration:

\[ z_{tt}(x, t) = z_{xx}(x, t) + f(x, t), \quad \forall x \in (0, 1), \ t \geq 0, \] (3)

Boundary conditions, \( \forall t \geq 0, \)

\[ z(0, t) = 0, \]
\[ z(1, t) = 0, \] (4)

and with the following initial condition, \( \forall x \in (0, 1), \)

\[ z(x, 0) = z^0(x), \]
\[ z_t(x, 0) = z^1(x), \] (5)

where \( z^0 \) and \( z^1 \) stand respectively for the initial deflection and the initial deflection speed.
When closing the loop with a linear control

Let us define the linear control by

\[ f(x, t) = -az_t(x, t), \ x \in (0, 1), \ \forall t \geq 0, \tag{6} \]

and consider

\[ V_1 = \frac{1}{2} \int (z_x^2 + z_t^2) \, dx. \]

Formal computation. Along the solutions to (3), (4) and (6):

\[
\begin{align*}
\dot{V}_1 &= \int_0^1 (z_x z_{xt} - a z_t^2 + z_t z_{xx}) \, dx \\
&= - \int_0^1 a z_t^2 \, dx + [z_t z_x]_{x=0}^{x=1} \\
&= - \int_0^1 a z_t^2 \, dx
\end{align*}
\]

Thus, it \( a > 0 \), \( V_1 \) is a (non strict) Lyapunov function.
Using standard technics (Lumer-Philipps theorem (for the well-posedness) and Huang-Prüss theorem (for the exp. stability)): 

\[ \forall a > 0, \forall (z^0, z^1) \text{ in } H^1_0(0,1) \times L^2(0,1), \]
\[ \exists ! \text{ solution } z: [0, \infty) \rightarrow H^1_0(0,1) \times L^2(0,1) \text{ to (3)-(6). Moreover, } \exists C, \mu > 0, \text{ such that, for any initial condition } H^1_0(0,1) \times L^2(0,1), \]
\[ \forall t \geq 0, \]
\[ \|z\|_{H^1_0(0,1)} + \|z_t\|_{L^2(0,1)} \leq Ce^{-\mu t}(\|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)}). \]

In the previous proposition:

- stability
- attractivity of the equilibrium
- with an exponential speed
Using standard technics (Lumer-Philipps thereom (for the well-posedness) and Huang-Prüss theorem (for the exp. stability)):

**Proposition**

\[ \forall a > 0, \forall (z^0, z^1) \text{ in } H^1_0(0, 1) \times L^2(0, 1), \]
\[ \exists \! \text{ solution } z: [0, \infty) \to H^1_0(0, 1) \times L^2(0, 1) \text{ to (3)-(6). Moreover, } \]
\[ \exists C, \mu > 0, \text{ such that, for any initial condition } H^1_0(0, 1) \times L^2(0, 1), \]
\[ \text{it holds, } \forall t \geq 0, \]
\[ \|z\|_{H^1_0(0,1)} + \|z_t\|_{L^2(0,1)} \leq Ce^{-\mu t}(\|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)}). \]

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\[ \exists ! \text{ solution } z: [0, \infty) \rightarrow H^1_0(0, 1) \times L^2(0, 1) \text{ to } (3)-(6). \]

Moreover,

\[ \exists C, \mu > 0, \text{ such that, for any initial condition } H^1_0(0, 1) \times L^2(0, 1), \]

it holds, \( \forall t \geq 0, \)

\[ \|z\|_{H^1_0(0, 1)} + \|z_t\|_{L^2(0, 1)} \leq Ce^{-\mu t}(\|z^0\|_{H^1_0(0, 1)} + \|z^1\|_{L^2(0, 1)}). \]

In the previous proposition:

- stability
- attractivity of the equilibrium
- with an exponential speed
When closing the loop with a saturating control

Let us consider now the nonlinear control

\[ f(x, t) = -\text{sat}(az_t(x, t)), \ x \in (0, 1), \ \forall t \geq 0, \]  

(7)

where \( \text{sat} \) is the localized saturated map:

\[ \text{sat}(\sigma) = \begin{cases} 
\sigma & \text{if } |\sigma| < 1 \\
\text{sign}(\sigma) & \text{else}
\end{cases} \]

Equation (3) in closed loop with the control (7) becomes

\[ z_{tt} = z_{xx} - \text{sat}(az_t) \]  

(8)

A formal computation gives, along the solutions to (8) and (4),

\[ \dot{V}_1 = -\int_0^1 z_t \text{sat}(az_t) dx \]

which asks to handle the nonlinearity \( z_t \text{sat}(az_t) \).
Remark: Choice of the saturation map

[Slemrod; 1989] and [Lasiecka and Seidman; 2003] deal with $L^2$ saturation:

Given $\sigma : [0, 1] \to \mathbb{R}$, $\text{sat}_2(\sigma)$ is the function defined by

$$\text{sat}_2(\sigma)(x) = \begin{cases} \frac{\sigma(x)}{\|\sigma\|_{L^2(0,1)}} & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\ \sigma(x) & \text{else} \end{cases}$$

Here we consider localized saturation which is more physically relevant:

$$\text{sat}(\sigma(x)) = \begin{cases} \sigma(x) & \text{if } |\sigma(x)| < 1 \\ \text{sign}(\sigma(x)) & \text{else} \end{cases}$$
Well-posedness of this nonlinear PDE

Theorem 1 [CP, Tarbouriech, Gomes da Silva Jr; 2016]

∀ a ≥ 0, for all (z^0, z^1) in \((H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1)\), there exists a unique solution \(z: [0, \infty) \rightarrow H^2(0, 1) \cap H^1_0(0, 1)\) to (8) with the boundary conditions (4) and the initial condition (5).
Consider

\[ A_1 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u_{xx} - \text{sat}(av) \end{pmatrix} \]

with the domain \( D(A_1) = (H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1) \).

Let us use a generalization of Lumer-Phillips theorem which is the so-called Crandall-Liggett theorem, as given in [Barbu; 1976]. See also [Brezis; 1973] and [Miyadera; 1992].

Again two conditions

1. \( A_1 \) is dissipative, that is

\[ \Re \left( \langle A_1 \begin{pmatrix} u \\ v \end{pmatrix} - A_1 \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \rangle \right) \leq 0 \]

2. For all \( \lambda > 0 \), \( D(A_1) \subset \text{Ran}(I - \lambda A_1) \)}
First item: Easy step!

Instead of proving
\[ \Re \left( \langle A_1 \left( \begin{array}{c} u \\ v \end{array} \right) - A_1 \left( \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right) , \left( \begin{array}{c} u \\ v \end{array} \right) - \left( \begin{array}{c} \tilde{u} \\ \tilde{v} \end{array} \right) \rangle \right) \leq 0, \]
let us check, for all \( \left( \begin{array}{c} u \\ v \end{array} \right) \in H^1_0(0, 1) \times L^2(0, 1) \):

\[ \Re \left( \langle A_1 \left( \begin{array}{c} u \\ v \end{array} \right) , \left( \begin{array}{c} u \\ v \end{array} \right) \rangle \right) \leq 0 \]

To do that, using the definition of \( A_1 \), and of the scalar product in \( H^1_0(0, 1) \times L^2(0, 1) \), it is equal to:

\[
\begin{align*}
&\int_0^1 v_x(x)u_x(x)dx + \int_0^1 (u_{xx}(x) - \text{sat}(a v(x)))v(x)dx, \\
&= \int_0^1 v_x(x)u_x(x)dx + \int_0^1 u_{xx}(x)v(x)dx - \int_0^1 \text{sat}(a v(x))v(x)dx \\
&= [u_x(x)v(x)]_{x=0}^{x=1} - \int_0^1 \text{sat}(a v(x))v(x)dx \leq 0
\end{align*}
\]

due to the boundary and since \( a \geq 0 \).
Second item asks to deal with a nonlinear ODE. Let \( \begin{pmatrix} u \\ v \end{pmatrix} \in H_0^1(0, 1) \times L^2(0, 1) \) we have to find \( \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \in D(A_1) \) such that

\[
(I - \lambda A_1) \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}
\]

that is

\[
\begin{cases}
\tilde{u} - \lambda \tilde{v} = u \\
\tilde{v} - \lambda (\tilde{u}_{xx} - \text{sat}(a \tilde{v})) = v
\end{cases}
\]

In particular, we have to find \( \tilde{u} \) such that

\[
\tilde{u}_{xx} - \frac{1}{\lambda^2} \tilde{u} - \text{sat} \left( \frac{a}{\lambda} (\tilde{u} - u) \right) = -\frac{1}{\lambda} v - \frac{1}{\lambda^2} u
\]

\( \tilde{u}(0) = \tilde{u}(1) = 0 \)

holds.

Nonhomogeneous nonlinear ODE with two boundary conditions
Lemma

If $a$ is nonnegative and $\lambda$ is positive, then there exists $\tilde{u}$ solution to

\begin{align*}
\tilde{u}_{xx} - \frac{1}{\lambda^2} \tilde{u} - \text{sat}(\frac{a}{\lambda}(\tilde{u} - u)) &= -\frac{1}{\lambda} \nu - \frac{1}{\lambda^2} u \\
\tilde{u}(0) &= \tilde{u}(1) = 0
\end{align*}

(9)

To prove this lemma, let us introduce the following map

\[ T : L^2(0, 1) \to L^2(0, 1), \quad y \mapsto z = T(y), \]

where $z = T(y)$ is the unique solution to

\begin{align*}
z_{xx} - \frac{1}{\lambda^2} z &= -\frac{1}{\lambda} \nu - \frac{1}{\lambda^2} u + \text{sat}(\frac{a}{\lambda}(y - u)) \\
z(0) &= z(1) = 0.
\end{align*}

Prove that $T$ is well defined and apply the Schauder fixed-point theorem (see e.g., [Coron; 2007]), to deduce that there exists $y$ such that $T(y) = y$

\[ \tilde{u} = y \text{ solves (9)} \]
Lemma

If \( a \) is nonnegative and \( \lambda \) is positive, then there exists \( \tilde{u} \) solution to

\[
\tilde{u}_{xx} - \frac{1}{\lambda^2} \tilde{u} - \text{sat}\left(\frac{a}{\lambda}(\tilde{u} - u)\right) = -\frac{1}{\lambda} v - \frac{1}{\lambda^2} u
\]
\[
\tilde{u}(0) = \tilde{u}(1) = 0
\] (9)

To prove this lemma, let us introduce the following map

\[
\mathcal{T} : L^2(0, 1) \rightarrow L^2(0, 1),
\]
\[
y \mapsto z = \mathcal{T}(y),
\]

where \( z = \mathcal{T}(y) \) is the unique solution to

\[
z_{xx} - \frac{1}{\lambda^2} z = -\frac{1}{\lambda} v - \frac{1}{\lambda^2} u + \text{sat}\left(\frac{a}{\lambda}(y - u)\right),
\]
\[
z(0) = z(1) = 0.
\]

Prove that \( \mathcal{T} \) is well defined and apply the Schauder fixed-point theorem (see e.g., [Coron; 2007]), to deduce that there exists \( y \) such that \( \mathcal{T}(y) = y \)

\( \tilde{u} = y \) solves (9)
Global asymptotic stability of this nonlinear PDE

Theorem 2

\( \forall a > 0, \) for all \((z^0, z^1) \) in \((H^2(0, 1) \cap H^1_0(0, 1)) \times H^1_0(0, 1)\), the solution to (8) with the boundary conditions (4) and the initial condition (5) satisfies the following stability property, \( \forall t \geq 0, \)

\[
\|z(\cdot, t)\|_{H^1_0(0, 1)} + \|z_t(\cdot, t)\|_{L^2(0, 1)} \leq \|z^0\|_{H^1_0(0, 1)} + \|z^1\|_{L^2(0, 1)},
\]

together with the attractivity property

\[
\|z(\cdot, t)\|_{H^1_0(0, 1)} + \|z_t(\cdot, t)\|_{L^2(0, 1)} \to 0, \quad \text{as } t \to \infty.
\]
Due to Theorem 1, the formal computation

\[ \dot{V}_1 = - \int_0^1 z_t \text{sat}(az_t) \, dx \]

makes sense. This is only a weak Lyapunov function \( \dot{V}_1 \leq 0 \)
(the state is \((z, z_t)\), and there is no \(-z^2\)).

To be able to apply LaSalle’s Invariance Principle, we have to check that the trajectories are precompact
(see e.g. [Dafermos, Slemrod; 1973], [d’Andréa-Novel et al; 1994]).

It comes from:

**Lemma**

The canonical embedding from \( D(A_1) \), equipped with the graph norm, into \( H^1_0(0, 1) \times L^2(0, 1) \) is compact.
Sketch of the proof of

The canonical embedding from $D(A_1)$, equipped with the graph norm, into $H_0^1(0,1) \times L^2(0,1)$ is compact.

Consider a sequence $\left( \begin{array}{c} u_n \\ v_n \end{array} \right)_{n \in \mathbb{N}}$ in $D(A_1)$, which is bounded with the graph norm, that is $\exists M > 0$, $\forall n \in \mathbb{N}$,

$$\left\| \left( \begin{array}{c} u_n \\ v_n \end{array} \right) \right\|^2_{D(A_1)} := \left\| \left( \begin{array}{c} u_n \\ v_n \end{array} \right) \right\|^2 + \left\| A_1 \left( \begin{array}{c} u_n \\ v_n \end{array} \right) \right\|^2,$$

$$= \int_0^1 \left( |u_n'|^2 + |v_n|^2 + |v_n'|^2 \right.$$

$$+ \left| u_n'' - asat(v_n) \right|^2)dx < M.$$

From that, we deduce that $\int_0^1 (|v_n|^2 + |v_n'|^2)dx$ and $\int_0^1 (|u_n'|^2 + |u_n''|^2)dx$ are bounded.

Thus there exists a subsequence which converges in $H_0^1(0,1) \times L^2(0,1).$
Using the dissipativity of $A_1$, and previous lemma the trajectory 
\[
( \begin{pmatrix} z(\cdot, t) \\ z_t(\cdot, t) \end{pmatrix} )
\]
is precompact in $H^1_0(0,1) \times L^2(0,1)$.

Moreover the $\omega$-limit set $\omega \left[ \begin{pmatrix} z(\cdot, 0) \\ z_t(\cdot, 0) \end{pmatrix} \right] \subset D(A_1)$, is not empty and invariant with respect to the nonlinear semigroup $T(t)$ (see [Slemrod; 1989]). We now use LaSalle’s invariance principle to show that

\[
\omega \left[ \begin{pmatrix} z(\cdot, 0) \\ z_t(\cdot, 0) \end{pmatrix} \right] = \{0\}.
\]

Therefore the convergence property holds.
Remark: Boundary control

1D wave equation with a boundary control.

**Dynamics:** \( \forall x \in (0,1), t \geq 0, \)

\[ z_{tt}(x, t) = z_{xx}(x, t), \]

**Boundary conditions:** \( \forall t \geq 0, \)

\[ z(0, t) = 0, \]
\[ z_x(1, t) = -\text{sat}(b z_t(1, t)), \]

In the same work, stability proof using the sector condition + strict Lyapunov function.
For the wave equation+ saturated in-domain control, a non-strict Lyapunov function has been computed. Thus:

- No robustness margin. What happens in presence of noise?
- For linear PDE, we have exponential convergence (see Proposition on Slide 10). Do we have exp. stability for the nonlinear PDE?

Let us start with the abstract control system

\[
\begin{align*}
\frac{d}{dt} z &= Az + Bu, \\
z(0) &= z_0,
\end{align*}
\]  

(10)

where \( A : D(A) \subset H \to H \) be a linear operator whose domain \( D(A) \) is dense in \( H \). Assume it generates a strongly continuous semigroup of contractions denoted by \( e^{tA} \). Let \( B : U \to H \) be a bounded operator. Wave equation with in-domain control applies!
A natural feedback law for (10) is \( u = -B^*z \).

Assumption 1: a linear feedback law is given

The linear closed-loop system

\[
\begin{aligned}
\frac{d}{dt} z &= (A - BB^*)z, \\
\end{aligned}
\]

globally exponentially stable.

Under Assumption 1, there exist a self-adjoint and definite positive operator \( P \in \mathcal{L}(H) \) and a positive value \( C \) such that

\[
\langle \tilde{A}z, Pz \rangle_H + \langle Pz, \tilde{A}z \rangle_H \leq -C\|z\|_H^2, \quad \forall z \in D(\tilde{A}),
\]

with \( \tilde{A} = A - BB^* \).
A natural feedback law for (10) is $u = -B^*z$.

**Assumption 1:** a linear feedback law is given

The linear closed-loop system

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\begin{aligned}
\frac{d}{dt} z &= (A - BB^*)z, \\
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(11)

globally exponentially stable.

Under Assumption 1, there exist a self-adjoint and definite positive operator $P \in \mathcal{L}(H)$ and a positive value $C$ such that

\[
\langle \tilde{A}z, Pz \rangle_H + \langle Pz, \tilde{A}z \rangle_H \leq -C\|z\|_H^2, \quad \forall z \in D(\tilde{A}),
\]  

(12)

with $\tilde{A} = A - BB^*$. 

Consider the **saturated** case

\[
\begin{align*}
\frac{d}{dt} z &= Az - B_{\text{sat}} U(B^* z), \\
z(0) &= z_0,
\end{align*}
\]
Consider the saturated case + disturbance

\[ \begin{cases} 
\frac{d}{dt}z = Az - B_{\text{sat}}U(B^*z + d), \\
 z(0) = z_0,
\end{cases} \]  

(13)

where \( d : (0, \infty) \to U \) is a disturbance.
In presence of saturated input and disturbances

Consider the saturated case + disturbance

\[
\begin{aligned}
\frac{d}{dt} z &= A z - B \text{sat}_U(B^* z + d), \\
z(0) &= z_0,
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\]

(13)

where \(d : (0, \infty) \rightarrow U\) is a disturbance.

Recall the \(L^2\) saturation: Given \(u : [0, 1] \rightarrow \mathbb{R}\), \(\text{sat}_2(\sigma)\) is the function defined by

\[
\text{sat}_2(\sigma) = \begin{cases} 
\sigma & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\
\frac{\sigma}{\|\sigma\|_{L^2(0,1)}} & \text{else}
\end{cases}
\]
In presence of saturated input and disturbances

Consider the saturated case + disturbance

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\begin{aligned}
\frac{d}{dt} z &= Az - B \text{sat}_U (B^* z + d), \\
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\end{cases}
\]

Here the same with \(U = L^2(0,1)\): Given \(\sigma \in U\), \(\text{sat}_U(\sigma)\) is the function defined by

\[
\text{sat}_U(\sigma) = \begin{cases} 
\sigma & \text{if } \|\sigma\|_U < 1 \\
\frac{\sigma}{\|\sigma\|_U} & \text{else}
\end{cases}
\]
In presence of saturated input and disturbances

Consider the saturated case + disturbance

\[
\begin{align*}
\frac{d}{dt} z &= A z - B \text{sat}_U (B^* z + d), \\
z(0) &= z_0,
\end{align*}
\]

where \( d : (0, \infty) \to U \) is a disturbance.

Recall the \( L^2 \) saturation: Given \( u : [0, 1] \to \mathbb{R} \), \( \text{sat}_2(\sigma) \) is the function defined by

\[
\text{sat}_2(\sigma) = \begin{cases} 
\sigma & \text{if } \|\sigma\|_{L^2(0,1)} < 1 \\
\sigma \|\sigma\|_{L^2(0,1)} & \text{else}
\end{cases}
\]

Here the same with \( U = L^2(0,1) \): Given \( \sigma \in U \), \( \text{sat}_U(\sigma) \) is the function defined by

\[
\text{sat}_U(\sigma) = \begin{cases} 
\sigma & \text{if } \|\sigma\|_U < 1 \\
\sigma \|\sigma\|_U & \text{else}
\end{cases}
\]

What can be said about the exp. stability when \( d = 0 \) and about the robustness in presence of \( d \)?
Input-to-State Stability definition

A positive definite function $V: H \rightarrow \mathbb{R}_{\geq 0}$ is said to be an ISS-Lyapunov function with respect to $d$ if $\exists$ two class $\mathcal{K}_\infty$ functions $\alpha$ and $\rho$ such that, for any solution to (13)

$$\frac{d}{dt} V(z) \leq -\alpha(\|z\|) + \rho(\|d\| u).$$

Remark: Of course ISS Lyapunov function
+ $\exists$ two functions $\alpha$ and $\bar{\alpha}$ of class $\mathcal{K}$ such that

$$\alpha(\|z\|_H) \leq V(z) \leq \bar{\alpha}(\|z\|_H), \forall z \in H$$

$\Rightarrow$ the origin of (13) with $d = 0$ is globally asymptotically stable.
Input-to-State Stability definition

A positive definite function $V : H \rightarrow \mathbb{R}_{\geq 0}$ is said to be an ISS-Lyapunov function with respect to $d$ if $\exists$ two class $\mathcal{K}_\infty$ functions $\alpha$ and $\rho$ such that, for any solution to (13)

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**Remark:** Of course ISS Lyapunov function

+ $\exists$ two functions $\underline{\alpha}$ and $\overline{\alpha}$ of class $\mathcal{K}$ such that

$$\underline{\alpha}(\|z\|_H) \leq V(z) \leq \overline{\alpha}(\|z\|_H), \forall z \in H$$

$\Rightarrow$ the origin of (13) with $d = 0$ is globally asymptotically stable.
Theorem 3 [Marx, Chitour, CP; 2018]

Suppose that Assumption 1 holds and let $P \in \mathcal{L}(H)$ be a self-adjoint and positive operator satisfying (12). Then, there exists $M$ such that

$$V(z) := \langle Pz, z \rangle_H + M\|z\|_H^3$$

(14)

is an ISS-Lyapunov function for (13).

The proof follows the finite-dimensional case considered in [Liu, Chitour, and Sontag; 1996].
Suppose that Assumption 1 holds and let $P \in \mathcal{L}(H)$ be a self-adjoint and positive operator satisfying (12). Then, there exists $M$ such that

$$V(z) := \langle Pz, z \rangle_H + M\|z\|_H^3$$

is an ISS-Lyapunov function for (13).

The proof follows the finite-dimensional case considered in [Liu, Chitour, and Sontag; 1996].
Let us consider the following candidate Lyapunov function

\[ V(z) := \langle Pz, z \rangle_H + \frac{2M}{3} \|z\|_H^3 = V_1(z) + \frac{2M}{3} \|z\|_H^3, \quad (15) \]

Along the strong solutions to (13), with \( \tilde{A} = A - BB^* \)

\[
\frac{d}{dt} V_1(z) = \langle Pz, Az \rangle_H + \langle PAz, z \rangle_H \\
+ \langle PB(\text{sat}_U(B^*z) - \text{sat}_U(B^*z + d)), z \rangle_H \\
+ \langle z, PB(\text{sat}_U(B^*z) - \text{sat}_U(B^*z + d)) \rangle_H
\]
Let us consider the following candidate Lyapunov function

\[ V(z) := \langle Pz, z \rangle_H + \frac{2M}{3} \|z\|^3_H = V_1(z) + \frac{2M}{3} \|z\|^3_H, \]

Along the strong solutions to (13), with \( \tilde{A} = A - BB^* \)

\[
\frac{d}{dt} V_1(z) = \langle Pz, \tilde{A}z \rangle_H + \langle P\tilde{A}z, z \rangle_H \\
+ \langle PB(B^*z - \text{sat}_U(B^*z)), z \rangle_H \\
+ \langle Pz, B(B^*z - \text{sat}_U(B^*z)) \rangle_H \\
+ \langle PB(\text{sat}_U(B^*z) - \text{sat}_U(B^*z + d)), z \rangle_H \\
+ \langle z, PB(\text{sat}_U(B^*z) - \text{sat}_U(B^*z + d)) \rangle_H
\]
Let us consider the following candidate Lyapunov function

$$V(z) := \langle Pz, z \rangle_H + \frac{2M}{3} \|z\|^3_H = V_1(z) + \frac{2M}{3} \|z\|^3_H,$$

(15)

Along the strong solutions to (13), with $\tilde{A} = A - BB^*$

$$\frac{d}{dt} V_1(z) = \langle Pz, \tilde{A}z \rangle_H + \langle P\tilde{A}z, z \rangle_H$$

$$+ \langle PB(B^*z - \text{sat}_U(B^*z)), z \rangle_H$$

$$+ \langle Pz, B(B^*z - \text{sat}_U(B^*z)) \rangle_H$$

$$+ \langle PB(\text{sat}_U(B^*z) - \text{sat}_U(B^*z + d)), z \rangle_H$$

$$+ \langle z, PB(\text{sat}_U(B^*z) - \text{sat}_U(B^*z + d)) \rangle_H$$

$$\leq -C\|z\|^2_H + 2\|B^*z\|_U \|P\|_{\mathcal{L}(H)} \|B^*z - \text{sat}_U(B^*z)\|_U$$

$$+ 2\langle \text{sat}_U(B^*z) - \text{sat}_U(B^*z + d), B^*Pz \rangle_U,$$
Let us consider the following candidate Lyapunov function

\[ V(z) := \langle Pz, z \rangle_H + \frac{2M}{3} \|z\|_H^3 = V_1(z) + \frac{2M}{3} \|z\|_H^3, \]  

(15)

Along the strong solutions to (13), with \( \tilde{A} = A - BB^* \)

\[
\frac{d}{dt} V_1(z) = \langle Pz, \tilde{A}z \rangle_H + \langle P\tilde{A}z, z \rangle_H \\
+ \langle PB(B^*z - \text{sat}_U(B^*z)), z \rangle_H \\
+ \langle Pz, B(B^*z - \text{sat}_U(B^*z)) \rangle_H \\
+ \langle PB(\text{sat}_U(B^*z) - \text{sat}_U(B^*z + d)), z \rangle_H \\
+ \langle z, PB(\text{sat}_U(B^*z) - \text{sat}_U(B^*z + d)) \rangle_H \\
\leq -C \|z\|_H^2 + 2\|B^*z\|_U \|P\|_{L(H)} \|B^*z - \text{sat}_U(B^*z)\|_U \\
+ 2\langle \text{sat}_U(B^*z) - \text{sat}_U(B^*z + d), B^*Pz \rangle_U, \\
\leq -C \|z\|_H^2 + 2\|B^*z\|_U \|P\|_{L(H)} \|B^*z - \text{sat}_U(B^*z)\|_U \\
+ 2k \|d\|_U \|B^*\|_{L(H,U)} \|P\|_{L(H)} \|z\|_H,
\]

using \( \text{sat}_U \) Lipchitz, Cauchy-Schwarz inequality and the fact that \( B^* \) is bounded.
Moreover using $\|d\|_U \|z\|_H \leq \varepsilon \|d\|^2_U + \frac{1}{\varepsilon} \|z\|^2_H$ and $\|B^*z - \text{sat}_U(B^*z)\|_U \leq \langle \text{sat}_U(B^*z), B^*z \rangle_U$, we get

$$\frac{d}{dt} V_1(z) \leq - \left( C - \frac{\|B^*\|_L(H,U) \|P\|_L(H)}{\varepsilon_1} \right) \|z\|^2_H$$

$$+ 2\|B^*\|_L(H,U) \|P\|_L(H) \|z\|_H \langle \text{sat}_U(B^*z), B^*z \rangle_U$$

$$+ k^2 \varepsilon_1 \|d\|^2_U$$

where $\varepsilon_1$ is a positive value that will be selected later.

Thus

$$\frac{d}{dt} V_1(z) \leq \text{good term} + \text{bad term} + d^2$$
Secondly, using the dissipativity of the operator $A_{\text{sat}}$, 
\[
\langle \text{sat}_U(B^*z) - \text{sat}_U(B^*z + d), B^*z \rangle_U \leq C_0 \|d\|_U, \text{ and}
\]
\[
\|z\|_H \|d\|_U \leq \frac{1}{\varepsilon_2} \|z\|_H^2 + \varepsilon_2 \|d\|_U^2,
\]
one has
\[
\frac{2M}{3} \frac{d}{dt} \|z\|_H^3 = M \|z\| \left( \langle Az, z \rangle_H + \langle z, Az \rangle_H \right)
\]
\[
- 2M \|z\|_H \langle B \text{sat}_U(B^*z + d), z \rangle_H
\]
\[
\leq - 2M \|z\|_H \langle \text{sat}_U(B^*z), B^*z \rangle_U
\]
\[
+ \langle \text{sat}_U(B^*z) - \text{sat}_U(B^*z + d), B^*z \rangle_U
\]
\[
\leq - 2M \|z\|_H \langle \text{sat}_U(B^*z), B^*z \rangle_U
\]
\[
+ 2MC_0 \|z\|_H \|d\|_U
\]
\[
\leq -2M \|z\|_H \langle \text{sat}_U(B^*z), B^*z \rangle_U
\]
\[
+ \frac{2MC_0}{\varepsilon_2} \|z\|_H^2 + 2MC_0 \varepsilon_2 \|d\|_U^2,
\]
where $\varepsilon_2$ is a positive value that has to be selected. For an appropriate choice of $M$, $\varepsilon_1$ and $\varepsilon_2$ we deduce the result. \qed
Secondly, using the dissipativity of the operator $A_{\text{sat}}$, 
\[ \langle \text{sat}_U (B^* z) - \text{sat}_U (B^* z + d), B^* z \rangle_U \leq C_0 \| d \|_U, \text{ and} \]
\[ \| z \|_H \| d \|_U \leq \frac{1}{\varepsilon_2} \| z \|_H^2 + \varepsilon_2 \| d \|_U^2, \]
one has
\[
\frac{2M}{3} \frac{d}{dt} \| z \|_H^3 = M \| z \| (\langle Az, z \rangle_H + \langle z, Az \rangle_H) \\
- 2M \| z \|_H \langle Bs_{\text{sat}} U (B^* z + d), z \rangle_H \\
\leq - 2M \| z \|_H (\langle \text{sat}_U (B^* z), B^* z \rangle_U \\
+ \langle \text{sat}_U (B^* z) - \text{sat}_U (B^* z + d), B^* z \rangle_U) \\
\leq - 2M \| z \|_H \langle \text{sat}_U (B^* z), B^* z \rangle_U \\
+ 2MC_0 \| z \|_H \| d \|_U \\
\leq - 2M \| z \|_H \langle \text{sat}_U (B^* z), B^* z \rangle_U \\
+ \frac{2MC_0}{\varepsilon_2} \| z \|_H^2 + 2MC_0 \varepsilon_2 \| d \|_U^2,
\]
where $\varepsilon_2$ is a positive value that has to be selected. For an appropriate choice of $M$, $\varepsilon_1$ and $\varepsilon_2$ we deduce the result. \qed
Secondly, using the dissipativity of the operator \( A_{\text{sat}} \),
\[
\langle \text{sat}_U(B^*z) - \text{sat}_U(B^*z + d), B^*z \rangle_U \leq C_0 \|d\|_U, \quad \text{and}
\]
\[
\|z\|_H \|d\|_U \leq \frac{1}{\varepsilon_2} \|z\|_H^2 + \varepsilon_2 \|d\|_U^2,
\]
once has
\[
2M \frac{d}{dt} \|z\|_H^3 = M \|z\| (\langle Az, z \rangle_H + \langle z, Az \rangle_H)
- 2M \|z\|_H \langle B \text{sat}_U(B^*z + d), z \rangle_H
\leq - 2M \|z\|_H (\langle \text{sat}_U(B^*z), B^*z \rangle_U
+ \langle \text{sat}_U(B^*z) - \text{sat}_U(B^*z + d), B^*z \rangle_U)
\leq - 2M \|z\|_H \langle \text{sat}_U(B^*z), B^*z \rangle_U
+ 2MC_0 \|z\|_H \|d\|_U
\leq -2M \|z\|_H \langle \text{sat}_U(B^*z), B^*z \rangle_U
+ \frac{2MC_0}{\varepsilon_2} \|z\|_H^2 + 2MC_0 \varepsilon_2 \|d\|_U^2,
\]
where \( \varepsilon_2 \) is a positive value that has to be selected. For an appropriate choice of \( M, \varepsilon_1 \) and \( \varepsilon_2 \) we deduce the result.
Secondly, using the dissipativity of the operator $A_{\text{sat}}$, 
\[ \langle \text{sat}_U(B^*z) - \text{sat}_U(B^*z + d), B^*z \rangle_U \leq C_0 \| d \|_U, \text{ and} \]
\[ \| z \|_H \| d \|_U \leq \frac{1}{\varepsilon_2} \| z \|^2_H + \varepsilon_2 \| d \|^2_U, \]
one has

\[
\frac{2M}{3} \frac{d}{dt} \| z \|^3_H = M \| z \| (\langle Az, z \rangle_H + \langle z, Az \rangle_H)
- 2M \| z \|_H \langle B \text{sat}_U(B^*z + d), z \rangle_H
\leq -2M \| z \|_H (\langle \text{sat}_U(B^*z), B^*z \rangle_U
+ \langle \text{sat}_U(B^*z) - \text{sat}_U(B^*z + d), B^*z \rangle_U)
\leq -2M \| z \|_H \langle \text{sat}_U(B^*z), B^*z \rangle_U
+ 2MC_0 \| z \|_H \| d \|_U
\leq -2M \| z \|_H \langle \text{sat}_U(B^*z), B^*z \rangle_U
+ \frac{2MC_0}{\varepsilon_2} \| z \|^2_H + 2MC_0 \varepsilon_2 \| d \|^2_U,
\]

where $\varepsilon_2$ is a positive value that has to be selected. For an
appropriate choice of $M$, $\varepsilon_1$ and $\varepsilon_2$ we deduce the result. \hfill \Box
Consider again the wave equation
(with S. Tarbouriech and JM Gomes da Silva; 16):

\[ z_{tt}(x, t) = z_{xx}(x, t) - \text{sat}(az_t(x, t)), \quad \forall x \in (0, 1), \quad t \geq 0, \]

**Boundary conditions**, \( \forall t \geq 0, \)

\[ z(0, t) = 0, \]
\[ z(1, t) = 0, \]

and with the following **initial condition**, \( \forall x \in (0, 1), \)

\[ z(x, 0) = z^0(x), \]
\[ z_t(x, 0) = z^1(x), \]

with \( z^0(x) = \sin(2\pi x) \) and \( z^1(x) = 0, \) for all \( x \in [0, 1]. \)
With the damping $a = 0.1$ and the level of the saturation $u_0 = 5$:

![Solution $z(x,t)$](image1)

![Saturating Input $\text{sat}(z_t(x,t))$](image2)

**Figure:** Time evolution of solution to nonlinear PDE with $u_0 = 5$.

**Figure:** Time evolution of the saturating input with $u_0 = 5$. 
Let us now select a lower saturation level: $u_0 = 1$. It takes a longer time to converge, but still converging!

**Figure**: Time evolution of solution to nonlinear PDE with $u_0 = 1$.  

**Figure**: Time evolution of the saturating input with $u_0 = 1$. 
Conclusion

- Well-posedness and global asymptotic stability for the nonlinear PDEs:

\[
\begin{aligned}
\begin{cases}
    z_{tt} = z_{xx} - \text{sat}(az_t) \\
    z(0, t) = z(1, t) = 0
\end{cases} & \quad \begin{cases}
    z_{tt} = z_{xx} \\
    z(0, t) = 0, \ z_x(1, t) = -\text{sat}(bz(1, t))
\end{cases}
\end{aligned}
\]

- control abstract problems and ISS results
- distributed/localized (localized/$L^2$) saturating control
- strict and non-strict Lyapunov functions have been used
Conclusion

- Well-posedness and global asymptotic stability for the nonlinear PDEs:

\[
\begin{align*}
    z_{tt} &= z_{xx} - \text{sat}(az_t) \\
    z(0, t) &= z(1, t) = 0 \\
    z(0, t) &= 0, \quad z_x(1, t) = -\text{sat}(bz(1, t))
\end{align*}
\]

- Control abstract problems and ISS results
- Distributed/localized (localized/$L^2$) saturating control
- Strict and non-strict Lyapunov functions have been used
**Under actual investigation**

**Under actual investigation #1**
ISS stability results for saturated boundary control?

**Under actual investigation #2**
Other PDE with saturated input? Beam equation? See also [Tarbouriech, CP, and Gomes da Silva Jr.; 2005] for anti-windup and (discretized) beam equation.

**Under actual investigation #3**
Anti-windup design to improve the performance?
**Under actual investigation**

**Under actual investigation #1**
ISS stability results for saturated boundary control?

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See also [Tarbouriech, CP, and Gomes da Silva Jr.; 2005] for anti-windup and (discretized) beam equation.

Under actual investigation #3
Anti-windup design to improve the performance?
1D wave equation with a boundary control.

Dynamics:

\[ z_{tt}(x, t) = z_{xx}(x, t), \quad \forall x \in (0, 1), \quad t \geq 0, \]  

(16)

Boundary conditions, \( \forall t \geq 0 \),

\[ z(0, t) = 0 , \]
\[ z_x(1, t) = g(t) , \]  

(17)

and with the same initial condition, \( \forall x \in (0, 1) \),

\[ z(x, 0) = z^0(x) , \]
\[ z_t(x, 0) = z^1(x) . \]  

(18)
When closing the loop with a linear boundary control

Let us define the linear control by

\[ g(t) = -bz_t(1, t), \ x \in (0, 1), \ \forall t \geq 0 \] (19)

and consider

\[ V_2 = \frac{1}{2} \int (e^{\mu x}(z_t + z_x)^2) + \int (e^{-\mu x}(z_t - z_x)^2) \]

Formal computation. Along the solutions to (16), (17) and (19):

\[ \dot{V}_2 = -\mu V_2 + \frac{1}{2} (e^{\mu}(1 - b)^2 - e^{-\mu}(1 + b)^2) z_t^2(1, t) \]

Assuming \( b > 0 \) and letting \( \mu > 0 \) such that \( e^{\mu}(1 - b)^2 \leq e^{-\mu}(1 + b)^2 \), it holds \( \dot{V}_2 \leq -\mu V_2 \) and thus \( V_2 \) is a strict Lyapunov function and thus (16)-(19) is exponentially stable.
When closing the loop with a linear boundary control

Let us define the linear control by

\[ g(t) = -bz_t(1, t), \ x \in (0, 1), \ \forall t \geq 0 \quad (19) \]

and consider

\[ V_2 = \frac{1}{2} \int (e^{\mu x}(z_t + z_x)^2 \, dx + \int (e^{-\mu x}(z_t - z_x)^2 \, dx, \]

**Formal computation.** Along the solutions to (16), (17) and (19):

\[ \dot{V}_2 = -\mu V_2 + \frac{1}{2} (e^{\mu}(1 - b)^2 - e^{-\mu}(1 + b)^2) z_t^2(1, t) \]

Assuming \( b > 0 \) and letting \( \mu > 0 \) such that \( e^{\mu}(1 - b)^2 \leq e^{-\mu}(1 + b)^2 \), it holds \( \dot{V}_2 \leq -\mu V_2 \) and thus \( V_2 \) is a strict Lyapunov function and thus (16)-(19) is exponentially stable.
When closing the loop with a saturating control

Let us consider now the nonlinear control

\[ g(t) = -\text{sat}(bz_t(1, t)), \forall t \geq 0. \]

The boundary conditions become:

\[ z(0, t) = 0, \quad z_x(1, t) = -\text{sat}(bz_t(1, t)). \] (20)

---

**Theorem 3**

\( \forall b > 0, \) for all \((z^0, z^1)\) in \(\{(u, v), (u, v) \in H^2(0, 1) \times H^1_0(0, 1), \ u_x(1) + \text{sat}(bv(1)) = 0, \ u(0) = 0\}\), the solution to (16) with the boundary conditions (20) and the initial condition (5) satisfies the following stability property, \( \forall t \geq 0, \)

\[
\|z(\cdot, t)\|_{H^1_0(0,1)} + \|z_t(\cdot, t)\|_{L^2(0,1)} \leq \|z^0\|_{H^1_0(0,1)} + \|z^1\|_{L^2(0,1)},
\]

together with the attractivity property

\[
\|z(\cdot, t)\|_{H^1_0(0,1)} + \|z_t(\cdot, t)\|_{L^2(0,1)} \to 0, \ \text{as} \ t \to \infty.
\]
To prove the well-posedness of the Cauchy problem we prove that $A_2$ defined by

$$A_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}$$

with the domain $D(A_2) = \{(u, v), (u, v) \in H^2(0, 1) \times H^1_{(0)}(0, 1), u'(1) + \text{sat}(bv(1)) = 0, u(0) = 0\}$ is a semigroup of contraction.

The global stability property comes directly from the dissipativity of $A_2$.

The global attractivity property comes from the following lemma:
To prove the well-posedness of the Cauchy problem we prove that $A_2$ defined by

$$A_2 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}$$

with the domain $D(A_2) = \{(u, v), (u, v) \in H^2(0, 1) \times H^1_{(0)}(0, 1), u'(1) + \text{sat}(bv(1)) = 0, u(0) = 0\}$ is a semigroup of contraction.

The global stability property comes directly from the dissipativity of $A_2$.

The global attractivity property comes from the following lemma:
Lemma (semi-global exponential stability)

For all $r > 0$, there exists $\mu > 0$ such that, for all initial condition satisfying

$$\|z^{0''}\|_{L^2(0,1)}^2 + \|z^1\|_{H^1_0(0,1)}^2 \leq r^2,$$  \hspace{1cm} (21)

it holds

$$\dot{V}_2 \leq -\mu V_2$$

along the solutions to (16) with the boundary conditions (20).
Sketch of the proof of this lemma

First note that by dissipativity of $A_2$, it holds that
\[ t \mapsto \left\| A_2 \begin{pmatrix} z(\cdot, t) \\ z_t(\cdot, t) \end{pmatrix} \right\| \]
is a non-increasing function. Thus, for all $t \geq 0$,
\[
|z_t(1, t)| \leq \left\| A_2 \begin{pmatrix} z(\cdot, 0) \\ z_t(\cdot, 0) \end{pmatrix} \right\|. \tag{22}
\]
Now for all initial conditions satisfying (21), there exists $c \neq b$ such that, for all $t \geq 0$,
\[
(b - c)|z_t(1, t)| \leq 1
\]
and thus the following local sector condition holds:
\[
\text{sat}(b \sigma) \quad \text{Letting } \sigma = z_t(1, t), \text{ it holds}
\]
\[
(\text{sat}(b \sigma) - b \sigma)(\text{sat}(b \sigma) - (b - c)\sigma) \leq 0
\]
We come back to the Lyapunov function candidate $V_2$. Given $b > 0$, using the previous inequality, we compute

$$
\dot{V}_2 = -\mu V_2 + e^\mu (\sigma - \text{sat}(b\sigma))^2 - e^{-\mu} (\sigma + \text{sat}(b\sigma))^2
\leq -\mu V_2 + \left( \begin{array}{c} \sigma \\ \text{sat}(b\sigma) \end{array} \right)^T \left( \begin{array}{cc} e^\mu - e^{-\mu} - b^2(b-c) & -e^\mu - e^{-\mu} + b + b(b-c) \\ -e^\mu - e^{-\mu} + b + b(b-c) & -1 + e^\mu - e^{-\mu} \end{array} \right) \left( \begin{array}{c} \sigma \\ \text{sat}(b\sigma) \end{array} \right)
\leq -\mu V_2
$$

with a suitable choice of constant values $\mu$ and $c$. The semi-global exponential stability follows. □
We come back to the Lyapunov function candidate $V_2$. Given $b > 0$, using the previous inequality, we compute

$$\dot{V}_2 = -\mu V_2 + e^\mu (\sigma - \text{sat}(b\sigma))^2 - e^{-\mu} (\sigma + \text{sat}(b\sigma))^2$$

$$\leq -\mu V_2 + \left( \begin{array}{c} \sigma \\ \text{sat}(b\sigma) \end{array} \right)^T \left( \begin{array}{cc} e^\mu - e^{-\mu} - b^2(b-c) & -e^\mu - e^{-\mu} + b + b(b-c) \\ -e^\mu - e^{-\mu} + b + b(b-c) & 1 + e^\mu - e^{-\mu} \end{array} \right) \left( \begin{array}{c} \sigma \\ \text{sat}(b\sigma) \end{array} \right)$$

$$\leq -\mu V_2$$

with a suitable choice of constant values $\mu$ and $c$. The semi-global exponential stability follows. □
We come back to the Lyapunov function candidate $V_2$. Given $b > 0$, using the previous inequality, we compute

$$
\dot{V}_2 = -\mu V_2 + e^\mu (\sigma - \text{sat}(b\sigma))^2 - e^{-\mu} (\sigma + \text{sat}(b\sigma))^2 \\
\leq -\mu V_2 + \left( \begin{array}{c} \sigma \\ \text{sat}(b\sigma) \end{array} \right)^T \left( \begin{array}{cc} e^\mu - e^{-\mu} - b^2 (b - c) & -e^\mu - e^{-\mu} + b + b(b - c) \\ -e^\mu - e^{-\mu} + b + b(b - c) & -1 + e^\mu - e^{-\mu} \end{array} \right) \left( \begin{array}{c} \sigma \\ \text{sat}(b\sigma) \end{array} \right) \\
\leq -\mu V_2
$$

with a suitable choice of constant values $\mu$ and $c$. The semi-global exponential stability follows. \qed