Contrôle et analyse de stabilité de systèmes de dimension infinie
Approches directes et indirectes par l'interpolation de Loewner

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Overview of the presentation

1. Introduction: Model Order Reduction (MOR) and Control
2. MOR-based control of infinite dimensional systems
3. MOR-based stability analysis
4. Conclusion
Motivations: why use model reduction?

Model approximation/reduction

Data
High/Infinite order model

Reduced-order model
\[ \dot{x} = Ax + Bu \]
\[ y =Cx + Du \]

- Simulation
- Analysis
- Optimization
- **Control**
Control of high-order systems

\[ P: \begin{align*}
    \dot{E} &= Ax + Bu \\
    y &= Cx + Du
\end{align*} \]

Output: \( e^{-\tau s} \)

Input: \( \frac{\partial y}{\partial t}(u, t) = \ldots \)

High-order system \( P \)
Control of high-order systems

\[
P: \begin{align*}
    Ex &= Ax + Bu \\
    y &= Cx + Du \\
    e^{-ts} &
\end{align*}
\]

Use \(P\)

High-order system \(P\)

High-order \(K\) tailored to \(P\)

Control of high-order systems

\[ P: \begin{align*}
    \dot{x} &= Ax + Bu \\
    y &= Cx + Du
\end{align*} \]

\[ e^{-\tau s} \]

\[ \frac{\partial y}{\partial t}(u, t) = \ldots \]

Use \( P \)

High-order \( K \) tailored to \( P \)

Use reduced-model \( P_r \)

\( K_r \), tailored to \( P_r \)

Control of high-order systems

\[ \dot{x} = Ax + Bu \\
y = Cx + Du \]

Use \( P \)

Use reduced-model \( P_r \)

High-order \( K \) tailored to \( P \)

\( K_r \) tailored to \( P_r \)

Model-based design
Control of high-order systems

Control of high-order systems

\[ P: \begin{align*}
\dot{x} &= Ax + Bu \\
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\end{align*} \]

Model reduction is everywhere

MOR-based control of infinite dimensional systems

1. Introduction: Model Order Reduction (MOR) and Control

2. MOR-based control of infinite dimensional systems
   - Illustrative example
   - The Loewner framework
   - Model-based approach
   - Data-driven approach
   - Model-based vs data-driven control

3. MOR-based stability analysis
   - Motivations
   - Loewner-based stability test

4. Conclusion
Illustrative example

\[
\frac{\partial \tilde{y}(x,t)}{\partial x} + 2x \frac{\partial \tilde{y}(x,t)}{\partial t} = 0 \quad \text{(transport equation)}
\]

\[
\tilde{y}(x,0) = 0 \quad \text{(initial condition)}
\]

\[
\tilde{y}(0, t) = \frac{1}{\sqrt{t}} \tilde{u}_f(0, t) \quad \text{(control input)}
\]

\[
\frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} u(0, s) = u_f(0, s) \quad \text{(controller bandwidth)},
\]
Illustrative example

\[ y(x, s) = \frac{\sqrt{\pi}}{\sqrt{s}} e^{-x^2 s} \frac{\omega_0^2}{s^2 + m\omega_0 s + \omega_0^2} u(s) = H(x, s)u(0, s) \]

\[ x = 1.9592 \]
The Loewner framework


Build $\hat{H}$ such that

$\forall k, \hat{H}(s_k) = H(s_k)$
The Loewner framework

The Loewner framework

The Loewner framework

Model-based approach

Input
Plant's data \( \{\omega_i, H(\omega_i)\}_{i=1}^{N} \)

Step 1
Obtain a reduced-order and rational model \( \hat{H} \) of the system \( H \)

Step 2
Design a controller \( C \)
Model-based approach

**Input**
Plant's data \( \{ \omega_i, H(\omega_i) \}_{i=1}^{N} \)

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Obtain a reduced-order and rational model \( \hat{H} \) of the system \( H \)

\( \forall i, \; \hat{H}(\omega_i) = H(\omega_i) \)

Loewner framework

**Step 2**
Design a controller \( C \)

**Graphs**
- Normalized singular value of the Loewner matrix
- Chosen order \( r = 33 \)
- Gain vs. Frequency
- \( H(s, x) \)
- \( \hat{H}(s, x) \), with \( r = 33 \)
Model-based approach

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Design a controller \( C \)

\[ C(s) = k_p + \frac{k_i}{s} \quad k_p = 0.191 \quad k_i = 0.0252 \]

\( \mathcal{H}_\infty \)-structured synthesis (hinfstruct)

![Graph showing \( \mathcal{H}_\infty \)-structured synthesis](image)

- Weight function \( W_e \)
- Sensitivity function \( \dot{S} \)
- Sensitivity function \( S \)

![Graph showing frequency response](image)

- Complementary sensitivity function \( \dot{M} \)
- Complementary sensitivity function \( M \)
Data-driven approach

\[ K^* = H^{-1}M(1 - M)^{-1} \]
Data-driven approach

**Input**
- Plant's data \(\{\omega_i, H(\omega_i)\}_{i=1}^{N}\)
- Reference model \(M\)

**Step 1**
Define the ideal controller \(K^*\)

**Step 2**
Obtain a reduced order controller \(K\)

\[
K^* = H^{-1}M(1-M)^{-1}
\]

Risk of instability compensation in the open-loop:

\[
\begin{align*}
H(0) &= \infty & M(0) &= 1 \\
H(\infty) &= 0 & M(\infty) &= 0
\end{align*}
\]
**Data-driven approach**

**Input**
- Plant's data \( \{\omega_i, H(\omega_i)\}_{i=1}^N \)
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\[
M_1(s) = \frac{1}{\frac{s^2}{\omega_0^2} + \frac{2s}{\omega_0} + 1} \\
M_2 = \frac{\hat{HC}}{1 + \hat{HC}}
\]

\( \omega_0 = 0.5 \text{rad} / s \)
Data-driven approach

- Plant's data \(\{\omega_i, H(\omega_i)\}_{i=1}^{N}\)
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\(\omega_0 = 0.5\text{rad/s}\)

**Step 2**
Obtain a reduced order controller \(K\)

\[
K_1(s) = \frac{0.1347s + 0.009259}{s + 0.001303}
\]

\[
K_2(s) = \frac{0.1914s + 0.02517}{s + 1.526 \cdot 10^{-5}} \approx C(s)
\]

Loewner framework
Data-driven approach

\[ K^* = H^{-1}M(1 - M)^{-1} \]

Risk of instability compensation in the open-loop

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\[ \omega_0 = 0.5 \text{ rad/s} \]
Model-based vs data-driven control

The Loewner framework can be used as a central tool for the control of infinite dimensional transfer functions

- Model based approach

- Data-driven approach
Model-based vs data-driven control

The Loewner framework can be used as a central tool for the control of infinite dimensional transfer functions

- **Model based approach**
  - more steps

- **Data-driven approach**
  + direct control design
  - less flexible specifications
Model-based vs data-driven control

The Loewner framework can be used as a central tool for the control of infinite dimensional transfer functions

- Model based approach
  - more steps
  - guaranteed stability but for the reduced-order model $\hat{H}$
- Data-driven approach
  + direct control design
  - less flexible specifications
  + conservative data-driven stability test
MOR-based stability analysis

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Motivations

Does the controller stabilise the real system?

Application of the small-gain theorem

The closed-loop is well-posed and internally stable for all stable $\Delta = K - K^*$ such that $\|\Delta\|_\infty \leq \gamma^{-1}$ if and only if $\|(1 - M)P\|_\infty < \gamma$

Motivations

Does the controller stabilise the real system?

Application of the small-gain theorem

The closed-loop is well-posed and internally stable for all stable \( \Delta = K - K^* \) such that \( \| \Delta \|_\infty \leq \gamma^{-1} \) if and only if \( \| (1 - M)P \|_\infty < \gamma \)

Loewner-based stability test

Loewner-based stability test

1. Compute samples $T(j\omega_i) = \frac{C(j\omega_i)H(j\omega_i)e^{-\tau j\omega_i}}{1+C(j\omega_i)H(j\omega_i)e^{-\tau j\omega_i}}$
Loewner-based stability test

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2. Obtain a minimal realisation $\hat{T}$ through the Loewner framework such that $\hat{T}(j\omega_i) = T(j\omega_i)$
Loewner-based stability test

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3. Compute $\hat{T}_s$

$$\hat{T}_s = \arg\min_{\hat{T} \in \mathcal{S}_{n,n_i,n_o}^{\dagger}} \|T - \hat{T}\|_{\infty}$$

On the closest stable descriptor system in the respective spaces $\mathcal{RH}_2$ and $\mathcal{RH}_\infty$, Kähler, M., Linear Algebra and its Applications, 2014.
Loewner-based stability test

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4. Compute the stability index as $S = \|\hat{T}_s - \hat{T}\|_\infty$
Loewner-based stability test

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   \]

4. Compute the stability index as $S = \|\hat{T}_s - \hat{T}\|_\infty$

   IF $S < \epsilon$ then $T$ is stable

   ELSE $S > \epsilon$ then $T$ is unstable
Results

Stability tag as a function of the delay $\tau$ in the loop.
Nyquist diagram for varying values of $\tau$: $S < 10^{-10}$ (stable configuration) and $S > 10^{-10}$ (unstable configuration).
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The Loewner framework is a versatile and efficient tool for the control of high/infinite-order systems.

- It provides a stability test when used with a projection on $\mathcal{RH}_\infty$.
- Move toward robustness analysis.

Model-based design:
- Use $P$
- High-order $K$ tailored to $P$

Data-driven design:
- Use reduced-model $P_r$
- $K_r$ tailored to $P_r$
- Use data
- Reduced controller
The Loewner framework is a versatile and efficient tool for the control of high/infinite-order systems.

- It provides a stability test when used with a projection on $\mathcal{RH}_\infty$
- Move toward robustness analysis
- Which frequencies to use? What about noise?