Structural properties of linear switched systems: observability, controllability, minimality

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Outline of the course

- Reminder: structural properties of linear systems.
- Observability of linear switched systems.
- Reachability/controllability of linear switched systems.
- Minimality of linear switched systems.
- Kalman-Ho realization algorithm
Linear Time Invariant (LTI) state-space representation

\[ \Sigma : \begin{cases} \sigma x(t) = Ax(t) + Bu(t) \\ y(t) =Cx(t) \end{cases} \]

\( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}. \)

\( \sigma x(t) = \begin{cases} \dot{x}(t) & \text{continuous time} \\ x(t+1) & \text{discrete time} \end{cases} \)

\((A, B, C)\): shorthand notation.
Observability: general definition

General non-linear system

\[ \sigma x(t) = f(x(t), u(t)), \ y = h(x(t), u(t)). \]

\( u \) – input, \( y \) – output.

\( y(z, u) \) – output signal from initial state \( z \) under input \( u \).

called observable (in the sense of indistinguishability), if

\[ \forall z_1 \neq z_2 : \exists u : y(z_1, u) \neq y(z_2, u) \]

i.e. for any two initial states \( z, z' \) there exists an input \( u(.) \) such that the corresponding outputs \( y, y' \) are different.

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i.e. for any two initial states \( z, z' \) for all inputs \( u(.) \) such that the corresponding outputs \( y, y' \) are different.
Observability: general definition

Observability in the sense of state reconstruction $\implies$ observability in the sense indistinguishability.

Observability in the sense of state reconstruction is necessary for observer design.

Observability in the sense of indistinguishability is necessary for minimal dimensional state-space representations.
Observability: linear case

\[ f(x, u) = Ax + Bu, \quad h(x, u) = Cx \]

Observability in the sense of state reconstruction \iff observability in the sense indistinguishability.

For linear systems

\[
\begin{align*}
y(z, u) &= y(z, 0) + y(0, u) \\
y(z', u) &= y(z', 0) + y(0, u) \\
y(z, u) &\neq y(z', u) \iff y(z, 0) + y(0, u) \neq y(z', 0) + y(0, u) \\
&\iff y(z, 0) \neq y(z', 0) \\
\exists u : y(z, u) &\neq y(z', u) \iff y(z, 0) \neq y(z', 0) \\
\forall u : y(z, u) &\neq y(z', u)
\end{align*}
\]
Observability rank condition

\((A, B, C)\) observable,\
\[
\iff \quad \text{rank} \left[ C^T \ A^T C^T \ \cdots \ \left( A^{n-1} \right)^T C^T \right]^T = n
\]
\[
\iff \quad \bigcap_{k=0}^{\infty} \ker CA^k = \{0\}
\]
\[
\iff \quad \forall z \neq 0, \exists k \geq 0 : 0 \neq CA^k z = \begin{cases} \frac{d^k}{dt^k} y(z, 0)(t)|_{t=0} & \text{cont. time} \\ y(z, 0)(t) & \text{disc. time} \end{cases}
\]
\[
\iff \quad \forall z \neq 0 : y(z, 0) \neq 0 \iff \forall z_1 \neq z_2 : y(z_1 - z_2, 0) \neq 0 \iff \forall z_1 \neq z_2 : y(z_1, 0) \neq y(z_2, 0).
\]
Observability: application

Observability \implies \text{existence of a Luenberger-observer}

Observability reduction: replace a system \((A, B, C)\) with another one with the same input-output behavior. Basis \(b_1, \ldots, b_n\) such that \(b_{o+1}, \ldots, b_n\) spans

\[
\text{ker} \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]

In the new basis

\[
A = \begin{bmatrix}
A_o & 0 \\
* & A_{uo}
\end{bmatrix}, \quad B = \begin{bmatrix}
B_o \\
*
\end{bmatrix}, \quad C = [C_o, \ 0]
\]

\((A_o, B_o, C_o)\) is observable, has the same input-output behavior as \((A, B, C)\).
Detour: input-output behaviors

Two different ways to view a system:

- System of equations: \( \sigma x(t) = f(x(t), u(t)), \quad y = h(x(t)) \)
- Set of observed input-output pairs \((y, u)\) (see ‘Behavioral approach’ by Jan C. Willems).

Input-output behavior of \( \sigma x(t) = f(x(t), u(t)), \quad y = h(x(t)) \)

\[
\mathcal{B}_{f,h} = \{(u, y) \mid \exists x : \sigma x(t) = f(x(t), u(t)), \ y = h(x(t))\}
\]

Input-output behavior is what we want to control, state-space representation is a tool for control synthesis.
Input-output behavior, input-output function, observability

Input-output function from initial state $x_0$:

$$I_{f,h,x_0} : u \mapsto y \text{ s.t. } \sigma x(t) = f(x(t), u(t)), \ y = h(x(t)), x(0) = x_0.$$  

Relationship between the two:

$$B_{f,h} = \bigcup_{x_0, u} \{ (I_{x_0,f,h(u)}, u) \}$$

Observability (in the sense of indistinguishability) $\iff$ the function $x_0 \mapsto I_{f,h,x_0}$ is one-to-one

Observability (in the sense of state reconstruction) $\iff$ for every $(u, y)$ there exists unique $x_0$ s.t. $I_{f,h,x_0}(u) = y$. 
Input-output behavior of linear systems

\[ I_{A,B,C,x_0}, \mathcal{B}_{A,B,C} - \text{input-output function } I_{f,h,x_0}/\text{input-output behavior } \mathcal{B}_{f,h}, f(x,u) = Ax + Bu, h(x,u) = Cx. \]

Nice properties:

\[ I_{A,B,C,x_0}(u) = I_{A,B,C,x_0}(0) + I_{A,B,C,0}(u) \]

- \[ I_{A,B,C,x_0}(0) = \begin{cases} \begin{align*} Ce^{At}x_0 & \text{cont. time} \\ CA^t x_0 & \text{disc. time} \end{align*} \end{cases} \] – depends (linearly) on the initial state, independent of input

- \[ I_{A,B,C,0}(u) = \begin{cases} \begin{align*} \int_0^t Ce^{A(t-s)}Bu(s)ds \\ \sum_{s=0}^{t-1} CA^{(t-s-1)}Bu(s) \end{align*} \end{cases} \] – depends (linearly) on the input, not on initial state.

\[ I_{A,B,C,0} \iff \text{transfer function } H(s) = C(sI - A)^{-1}B. \]
Transfer function are often identified with input-output behavior.

But: Transfer functions do not capture all the input-output behavior.

Transfer functions capture the input-output behavior which we can control and observe.
Motivating example

Consider two linear systems

\[
\sigma x = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 2 & 1 \end{bmatrix} x
\]

\[
\sigma x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} -2 & 1 \end{bmatrix} x
\]

They have the same input-output behavior from zero initial state (transfer functions are the same).

Yet, \( u = -2y \) stabilizes the first system, and not the second.

What is the problem? Which model is the wrong one?
Is \((A, B, C)\) below observable?

\[
A = \begin{bmatrix}
-3 & 0 & -1 \\
0 & -3 & 0 \\
1 & 0 & 1
\end{bmatrix}, \quad
B = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad
C = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}^T
\]

Use the definition and the rank condition to motivate your answer.
If it is not observable, find two states \(z, z'\) s.t. \(I_{A,B,C,z} = I_{A,B,C,z'}\).
Perform observability reduction.
Consider two linear systems

\[ \sigma x = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]

\[ y = \begin{bmatrix} 2 & 1 \end{bmatrix} x \]

\[ \sigma x = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \]

\[ y = \begin{bmatrix} -2 & 1 \end{bmatrix} x \]

Do they have the same input-output function from the zero initial state?

Do they have the same input-output behavior?
Observability reduction \((A_o, B_o, C_o)\) revisited

Correspondence between input-output functions

\[
l_{A_o,B_o,C_o,\mathcal{P}x_0} = l_{A,B,C,x_0}, \quad \mathcal{P} = \begin{bmatrix} l_o \\ 0 \end{bmatrix}.
\]

Transfer functions of \((A_o, B_o, C_o)\) and \((A, B, C)\) are equal:

\[
l_{A_o,B_o,C_o,0} = l_{A,B,C,0}
\]

The set of input-output functions (hence the input-output behavior) are preserved by observability reduction:

\[
\bigcup_{x_0} l_{A_o,B_o,C_o} = \bigcup_{x_0} l_{A,B,C,x_0}, \quad \mathcal{B}_{A,B,C} = \mathcal{B}_{A_o,B_o,C_o}.
\]

Control synthesis can be done on \((A_o, B_o, C_o)\) instead of \((A, B, C)\) (attention, unstable unobserved modes !).
Reachability & controllability

\( x(z, u)(t) \) – state of \((A, B, C)\) at time \(t\), under input \(u\), initial state \(z\).

A state \(z\), is **reachable from** \(x_0\), if \(z = x(x_0, u)(T)\) for some \(u\) and \(T\).

\((A, B, C)\) is **reachable**, if all states are reachable from \(0\).

\((A, B, C)\) is **controllable**, if for any \(z, z'\), there exists \(u\) and \(T\) s.t.
\[ x(z, u)(T) = z'. \]
Conditions for reachability

\[(A, B, C) \text{ reachable, } \iff \text{rank } \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n \]

\[\iff \text{Span}\{A^kBu \mid k \geq 0, u \in \mathbb{R}^m\} = n \]

\[\iff (A^T, C^T, B^T) \text{ is observable.} \]

Controllability (in cont. time or in disc. time if \(A\) is invertible) \(\iff\) reachability.

\[\text{Span}\{A^kBu \mid k \geq 0, u \in \mathbb{R}^m\} \text{ set of reachable states } x(0, u)(t) \text{ from zero.} \]
Conditions for reachability

Main idea:

▶ $\text{Span}\{A^k Bu \mid k \geq 0, u \in \mathbb{R}^m\}$ is the smallest vector space which contains states reachable from zero.

▶ The set of states reachable from zero is a vector space.

The proof of the equivalence of controllability and reachability is difficult: if a state can be reached from zero, then zero can be reached from that state.
Reachability reduction

Basis $b_1, \ldots, b_n$ such that $b_1, \ldots, b_r$ spans

$$\text{Im}[B \ AB \ \cdots \ A^{n-1}B]$$

In the new basis

$$A = \begin{bmatrix} A_r & \ast \\ 0 & A_{uc} \end{bmatrix}, \quad B = \begin{bmatrix} B_r \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_r & \ast \end{bmatrix}$$

$(A_r, B_r, C_r)$ is reachable, and has the same input-output function from the zero initial state as $(A, B, C)$

$$I_{A_r, B_r, C_r, 0} = I_{A, B, C, 0}$$

A state which is not reachable from zero cannot be influenced by inputs.
Reachability reduction preserves the input-output function generated by initial state 0.

It is not true that \((A, B, C)\) has the same input-output behavior as \((A_r, B_r, C_r)\).

When replacing \((A, B, C)\) by \((A_r, B_r, C_r)\), we lose behavior which cannot be controlled.
Reachability & controllability

Perform reachability reduction on

\[ \dot{x} = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u \]

\[ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \]

and

\[ \dot{x} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u \]

\[ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \]

Calculate the input-output functions of the reduced systems from 0

Calculate the input-output functions of the original systems from \([0, 1]^T\).
Transforming an LTI to a minimal one

Minimization procedure

1. Transform \((A, B, C)\) to a reachable \((A_r, B_r, C_r)\) with the same input-output function from the initial state zero.

2. Transform \((A_r, B_r, C_r)\) to an observable \((A_m, B_m, C_m)\) with the same input-output function from the initial state zero.

\[ (A_m, B_m, C_m) \text{ is reachable and observable, its input-output function from zero is the same as } (A, B, C). \]

\[ I_{A_m,B_m,C_m,0} = I_{A,B,C,0} \]

\[ \text{Dimension of } (A_m, B_m, C_m) \text{ is the smallest among all } (A', B', C') \text{ s.t.} \]

\[ I_{A',B',C',0} = I_{A,B,C,0} \]
Minimality

Let \( I \) be an input-output function.

1. \((A, B, C)\) is a minimal dimensional system such that 
   \( I_{A,B,C,0} = I \iff (A, B, C) \) is reachable from zero, and 
   \((A, B, C)\) is observable.

2. If \((A, B, C)\) and \((\hat{A}, \hat{B}, \hat{C})\) are minimal dimensional s.t. 
   \( I_{A,B,C,0} = I_{\hat{A},\hat{B},\hat{C},0} = I \) then they are isomorphic: 
   there exists a nonsingular matrix \( T \) s.t.: 
   \[
   TAT^{-1} = \hat{A}, \quad TB = \hat{B}, \quad CT^{-1} = \hat{C}.
   \]
Consequences of minimality for control

- If two reachable and observable LTI systems have the same transfer function, then they are isomorphic and have the same input-output behavior.

Transfer functions capture the input-output behavior of reachable and observable systems.

- Minimal LTI system which with the same transfer function isomorphic $\implies$ control design does not depend on the choice of the LTI state-space representation.

- Minimal LTI representations are observable & controllable: observer design and stabilization is always possible.

- Unobservable/uncontrollable eigenvalues are the only potential source of problems.

- Try to use minimal systems for control.

- Further applications: system identification, model reduction.
Definition of linear switched systems

\[ \sigma x(t) = f(x(t), u(t)), \quad y(t) = h(x(t), u(t)) \]
\[ f(x, u) = A_q x + B_q v, \quad u = (q, v) \]
\[ h(x, u) = C_q x, \quad u = (q, v) \]

Inputs \( u = (q, v) \)
\( q \in Q = \{1, 2, \ldots, d\} \) – discrete mode, \( v \) – continuous input

Outputs
\( y \) – continuous output

Dimension – \( n \), the dimension of the state \( x(t) \).

Linear switched systems: simplest class of hybrid systems.
\( \{A_q, B_q, C_q\}_{q \in Q} \) – shorthand notation.
Expressions for the state and output

Discrete-time

\[x(x_0, (q, v))(t) = A_{q(t-1)} \cdots A_{q(0)}x_0 + \sum_{k=0}^{t-1} A_{q(t-1)} \cdots A_{q(k+1)}B_{q(k)}v(k)\]

\[y(x_0, (q, v))(t) = C_{q(t)}A_{q(t-1)} \cdots A_{q(0)}x_0 + \sum_{k=0}^{t-1} C_{q(t)}A_{q(t-1)} \cdots A_{q(k+1)}B_{q(k)}v(k)\]
Expressions for the state and output

Continuous-time: \( q(s) = q_i \) for \( s \in \left[ \sum_{j=1}^{i-1} t_j, \sum_{j=1}^i t_j \right) \), \( t_j \geq 0 \), \( t = \sum_{j=1}^k t_j \).

\[
\begin{align*}
x(x_0, (q, v))(t) &= e^{A_{q_k} t_k} \ldots e^{A_{q_1} t_1} x_0 + \\
& \sum_{i=1}^k \int_0^{t_i} e^{A_{q_k} t_k} \ldots e^{A_{q_i} (t_i-s)} B_{q_i} u(s + \sum_{j=1}^{i-1} t_j) \\
y(x_0, (q, v))(t) &= C_{q_k} e^{A_{q_k} t_k} \ldots e^{A_{q_1} t_1} x_0 + \\
& \sum_{i=1}^k \int_0^{t_i} C_{q_k} e^{A_{q_k} t_k} \ldots e^{A_{q_i} (t_i-s)} B_{q_i} u(s + \sum_{j=1}^{i-1} t_j)
\end{align*}
\]
Expressions for state and output trajectories

\[ Q = \{1, 2, 3\} \]

Discrete-time: \( q(0) = 1, \ q(1) = 2, \ q(2) = 1, \ q(3) = 2 \). Write \( x(t), y(t) \) for \( t = 0, 1, 2, 3 \).

Continuous-time: \( k = 4, \ q_1 = 1, \ q_2 = 2, \ q_3 = 1, \ q_4 = 2 \). Write \( x(t), y(t) \).
Why structure theory of linear switched systems difficult

Local structure of LTI models does not determine the structure of the switched system.
Two modes: \( Q = \{1, 2\} \)

\[
A_1 = \begin{bmatrix}
-3 & 0 & -1 \\
0 & -3 & 0 \\
1 & 0 & 1
\end{bmatrix}, \quad
B_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad
C_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^T
\]

\[
A_2 = \begin{bmatrix}
-4 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & -1
\end{bmatrix}, \quad
B_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad
C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T
\]

The local subsystems are not observable, but the switched system is (we will see it later).
Observability of linear switched systems

\[ l\{A_q, B_q, C_q\}_q \in Q, x_0 \] is an input-output function \( l_{f, h, x_0}, f(x, (q, v)) = A_qx + B_qu, h(x, (q, v)) = C_qx. \)

\{A_q, B_q, C_q\}_q \in Q is observable, if the function \( x_0 \mapsto l\{A_q, B_q, C_q\}_q \in Q, x_0 \) is one-to-one.

Decomposition into autonomous and continuous input-dependent part:

\[
l\{A_q, B_q, C_q\}_q \in Q, x_0((q, v)) = \\
l\{A_q, B_q, C_q\}_q \in Q, x_0((q, 0)) + l\{A_q, B_q, C_q\}_q \in Q, 0((q, v))
\]

Exercise: Write down the analytic expressions for
\[ l\{A_q, B_q, C_q\}_q \in Q, x_0((q, 0)) \] and \[ l\{A_q, B_q, C_q\}_q \in Q, x_0((q, v)) \] (discrete or cont. time)
Theorem (Sun & Ge & Lee)

\{ A_q, B_q, C_q \}_{q \in Q} \text{ is observable, } \iff \n = \text{rank } \left[ (C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1})^T \mid q, q_1, \ldots, q_k \in Q, 0 \leq k < n \right] \iff \bigcap_{k=0}^{\infty} \bigcap_{q, q_1, \ldots, q_k \in Q} \ker C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1} = \{0\}
Condition for observability

A non-obvious fact from [Sun & Ge & Lee]:

\[
\bigcap_{k=0}^{\infty} \bigcap_{q, q_1, \ldots, q_k \in Q} \ker C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1} = \\
\bigcap_{k=0}^{n-1} \bigcap_{q, q_1, \ldots, q_k \in Q} \ker C_q A_{q_k} A_{q_{k-1}} \cdots A_{q_1}.
\]

Corollary

If for some q, \((C_q, A_q)\) is an observable pair, then \(\{A_q, B_q, C_q\}_{q \in Q}\) is observable.

Proof: Exercise
Observability of linear switched systems

\[ \{ A_q, B_q, C_q \}_{q \in Q} \text{ is observable, if } \forall x_0, x_0' : \]

\[ \forall q : I_{\{ A_q, B_q, C_q \}_{q \in Q}, x_0}(q, 0) = I_{\{ A_q, B_q, C_q \}_{q \in Q}, x_0'}(q, 0) \implies x_0 = x_0', \]

i.e., different initial states can be distinguished by the outputs for zero continuous input and some switching signal.

\[ I_{\{ A_q, B_q, C_q \}_{q \in Q}, x_0}(q, 0) \text{ linear in } x_0 \implies \]

\[ I_{\{ A_q, B_q, C_q \}_{q \in Q}, x_0}(q, 0) = I_{\{ A_q, B_q, C_q \}_{q \in Q}, x_0'}(q, 0) \iff \]

\[ I_{\{ A_q, B_q, C_q \}_{q \in Q}, x_0 - x_0'}(q, 0) = 0 \]

\[ \{ A_q, B_q, C_q \}_{q \in Q} \text{ is observable, if } \]

\[ (\forall q : I_{\{ A_q, B_q, C_q \}_{q \in Q}, x_0}(q, 0) = 0) \implies x_0 = 0. \]
Observability of linear switched systems

$$(\forall q \in Q : \{A_q, B_q, C_q\}_{q \in Q}, x_0((q, 0)) = 0) \iff C_qA_{q_k}A_{q_{k-1}} \cdots A_{q_1}x_0 = 0, \ \forall k \geq 0, q, q_1, \ldots, q_k \in Q$$
Observability: exercise

Two modes: \( Q = \{1, 2\} \)

\[
A_1 = \begin{bmatrix}
-3 & 0 & -1 \\
0 & -3 & 0 \\
1 & 0 & 1
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}^T
\]

\[
A_2 = \begin{bmatrix}
-4 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & -1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}^T
\]

Check observability
Observability: exercise

\[ Q = \{1, 2\} \]

\[ A_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \]

\[ B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \]

\[ C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \]

Check observability.
Observability reduction

\[ \mathcal{W}^* = \bigcap_{k=0}^{n-1} \bigcap_{q, q_1, \ldots, q_k \in Q} \ker C_q A_q A_{q-1} \cdots A_1 = \]

\[ \bigcap_{k=0}^{\infty} \bigcap_{q, q_1, \ldots, q_k \in Q} \ker C_q A_q A_{q-1} \cdots A_1. \]

\[ b_1, \ldots, b_n \text{ basis s.t. } b_{o+1}, \ldots, b_n \text{ span } \mathcal{W}^*. \]

In this new basis,

\[ A_q = \begin{bmatrix} A_q^O & 0 \\ A_q & A_q'' \end{bmatrix}, \quad C_q = \begin{bmatrix} C_q^O & 0 \end{bmatrix}, \quad B_q = \begin{bmatrix} B_q^O \\ B_q' \end{bmatrix}, \]
Observability reduction

\{A_q^O, B_q^O, C_q^O\}_{q \in Q} is observable.

The input-output behavior of \{A_q^O, B_q^O, C_q^O\}_{q \in Q} and \{A_q, B_q, C_q\}_{q \in Q} are the same.

\[ I\{A_q^O, B_q^O, C_q^O\}_{q \in Q}, P x_0 = I\{A_q, B_q, C_q\}_{q \in Q}, x_0. \]

\[ P = \begin{bmatrix} l_o & 0 \\ 0 & 0 \end{bmatrix} \]

Last \(n - o\) coordinates: unobservable part, does not influence the output, cannot be estimated from the output.
Observability: exercise

\[ A_{q_1} = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, B_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C_{q_1} = \begin{bmatrix} 1 \end{bmatrix}^T \]

\[ A_{q_2} = \begin{bmatrix} -4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, B_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, C_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \]

Perform observability reduction.
Reachability of linear switched systems

\( x(z, q, v)(t) \) – state of \( \{A_q, B_q, C_q\}_{q \in Q} \) at time \( t \), under input \( v \), switching signal \( q \), and initial state \( z \).

A state \( z \), is called \textbf{reachable from} \( x_0 \), if \( z = x(x_0, u)(T) \) for some \( u \) and \( T \).

\( \{A_q, B_q, C_q\}_{q \in Q} \) is called \textbf{reachable from} \( x_0 \), if all states are reachable from \( x_0 \).

\( \{A_q, B_q, C_q\}_{q \in Q} \) is called \textbf{span-reachable from} \( x_0 \), if the linear span of all states reachable from zero is the whole state-space.

\( \{A_q, B_q, C_q\}_{q \in Q} \) is called \textbf{controllable}, if for any \( z, z' \), there exists \( u \) and \( T \) s.t. \( x(z, u)(T) = z' \).
Reachability of linear switched systems

Theorem (Sun & Ge & Lee)
\{A_q, B_q, C_q\}_{q \in Q} is span-reachable from 0, ⇐⇒

\[ n = \text{rank} \left[ A_{q_k} A_{q_{k-1}} \cdots A_{q_1} B_{q_0} \mid q_0, q_1, \ldots, q_k \in Q, k < n \right] \]

⇐⇒

\[ n = \text{dim} \text{Span}\{ A_{q_k} A_{q_{k-1}} \cdots A_{q_1} B_{q_0} v \mid q_0, q_1, \ldots, q_k \in Q, k \geq 0, v \} \]

In continuous time or in discrete-time if \( A_q \) are invertible, then

- span reachability from 0 is equivalent to reachability from 0,
- reachability from 0 is equivalent to controllability.
Reachability of linear switched systems

A non-obvious fact from [Sun & Ge & Lee]:

\[
\text{Span}\{A_{q_k} A_{q_{k-1}} \cdots A_{q_1} B_{q_0} \nu \mid q_0, q_1, \ldots, q_k \in Q, k \geq 0, \nu \in \mathbb{R}^m\} = \\
\text{Span}\{A_{q_k} A_{q_{k-1}} \cdots A_{q_1} B_{q_0} \nu \mid q_0, q_1, \ldots, q_k \in Q, n > k \geq 0, \nu \in \mathbb{R}^m\}
\]

Corollary

If for some \( q \), \((A_q, B_q)\) is a controllable pair, then \( \{A_q, B_q, C_q\}_{q \in Q} \) is span-reachable from 0.

Proof: Exercise
Reachability of linear switched systems

Main idea:

$\text{Span}\{A_{q_k}A_{q_{k-1}} \cdots A_{q_1}B_{q_0}v \mid q_0, q_1, \ldots, q_k \in Q, k \geq 0, v \in \mathbb{R}^m\}$

is the smallest vector space which contains states reachable from zero.

In continuous time or in discrete-time if $A_q$ are invertible, then there exists a switching signal $q$ and an interval $[0, T]$ s.t.

- The linear span of

$$\{x(0, (q, v))(t) \mid v \text{ continuous input, } t \in [0, T]\}$$

contains the set of all states which are reachable from zero.

- The set

$$\{x(0, (q, v))(t) \mid v \text{ continuous input, } t \in [0, T]\}$$

is a vector space.

The proof of the equivalence of controllability and reachability is difficult.
Reachability: exercise

Two modes: \( Q = \{1, 2\} \)

\[
A_1 = \begin{bmatrix}
-3 & 0 & -1 \\
0 & -3 & 0 \\
1 & 0 & 1
\end{bmatrix}, \quad B_1 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad C_1 = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}^T
\]

\[
A_2 = \begin{bmatrix}
-4 & 0 & 1 \\
0 & -2 & 0 \\
1 & 0 & -1
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
1 \\
0 \\
1
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}^T
\]

Check reachability
Reachability: exercise

Q = \{1, 2\}

\[
A_1 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
A_2 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
B_2 = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix},
\]

\[
C_1 = C_2 = \begin{bmatrix}
1 & 0 & 0
\end{bmatrix},
\]

Check reachability.
Reachability reduction

\[ \mathcal{V}^* = \ \text{Span}\{A_{q_k}A_{q_{k-1}} \cdots A_{q_1}B_{q_0}\nu \mid q_0, q_1, \ldots, q_k \in Q, k \geq 0, \nu \in \mathbb{R}^m\} \]

Choose a basis \( b_1, \ldots, b_n \) s.t. \( b_1, \ldots, b_r \) span \( \mathcal{V}^* \).

In this new basis,

\[
A_q = \begin{bmatrix} A_q^R & A_q' \\ 0 & A_q'' \end{bmatrix}, \quad C_q = \begin{bmatrix} C_q^R & C_q' \end{bmatrix}, \quad B_q = \begin{bmatrix} B_q^R \\ 0 \end{bmatrix}, \quad (1)
\]

\( \{A_q^R, B_q^R, C_q^R\}_{q \in Q} \) is span-reachable from 0.

The input-output function from zero of \( \{A_q^R, B_q^R, C_q^R\}_{q \in Q} \) and \( \{A_q, B_q, C_q\}_{q \in Q} \) are the same.

\[
I_{\{A_q^R, B_q^R, C_q^R\}_{q \in Q}, 0} = I_{\{A_q, B_q, C_q\}_{q \in Q}, 0}.
\]

Last \( n - r \) coordinates: uncontrollable part, cannot be influenced by continuous inputs.
Reachability: exercise

\[ A_{q_1} = \begin{bmatrix} -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}, \quad B_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_{q_1} = \begin{bmatrix} 1 \end{bmatrix}^T \]

\[ A_{q_2} = \begin{bmatrix} -4 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}, \quad B_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \]

Apply reachability reduction.
Minimization

- Apply reachability reduction to \( \{A_q, B_q, C_q\}_{q \in Q} \) to get \( \{A^R_q, B^R_q, C^R_q\}_{q \in Q} \).

- Apply observability reduction to \( \{A^R_q, B^R_q, C^R_q\}_{q \in Q} \) to get \( \{A^m_q, B^m_q, C^m_q\}_{q \in Q} \).

\( \{A^m_q, B^m_q, C^m_q\}_{q \in Q} \) is span-reachable from 0, observable, and its input-output function from 0 is the same as that of \( \{A_q, B_q, C_q\}_{q \in Q} \), i.e.,

\[
I_{\{A^m_q, B^m_q, C^m_q\}_{q \in Q}, 0} = I_{\{A_q, B_q, C_q\}_{q \in Q}, 0}.
\]

State-space dimension of \( \{A^m_q, B^m_q, C^m_q\}_{q \in Q} \) is \( \leq \) state-space dimension of \( \{A_q, B_q, C_q\}_{q \in Q} \).
Minimality

Let \( I \) be an input-output function.

**Theorem (Pet06,Pet07,Pet11a,Pet13)**

\[
\{A_q, B_q, C_q\}_{q \in Q} \text{ is a minimal dimensional among all linear switched systems whose input-output function from } 0 \text{ is } I, \\
\iff \\
\{A_q, B_q, C_q\}_{q \in Q} \text{ is observable and span-reachable from } 0.
\]

\[
\{A^m_q, B^m_q, C^m_q\}_{q \in Q} \text{ is minimal dimensional among all linear switched systems with the same input-output function from } 0.
\]

\[
\text{If } \{A_q, B_q, C_q\}_{q \in Q} \text{ and } \{\hat{A}_q, \hat{B}_q, \hat{C}_q\}_{q \in Q} \text{ are minimal dimensional s.t. } I\{A_q, B_q, C_q\}_{q \in Q, 0} = I\{\hat{A}_q, \hat{B}_q, \hat{C}_q\}_{q \in Q, 0} = I \text{ then they are isomorphic: } \\
\forall q : TA_q T^{-1} = \hat{A}_q, \ TB_q = \hat{B}_q, \ CT_q^{-1} = \hat{C}_q.
\]
Counter-examples

- If at least one of the continuous subsystems are minimal, then the switched system is minimal.

- A switched system can be minimal (resp. observable, reachable), without any of the subsystems being minimal (resp. observable, reachable).

- Certain linear switched systems can never be brought to a form where all the continuous subsystems are minimal.
Example

\[
A_{q_1} = \begin{bmatrix}
-3 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -3
\end{bmatrix}, \quad B_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_{q_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^T
\]

\[
A_{q_2} = \begin{bmatrix}
-4 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -2
\end{bmatrix}, \quad B_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_{q_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^T
\]

This system is neither observable nor reachable, hence it is not minimal.
After minimization, we obtain

$$A_{q_1} = \begin{bmatrix} -3 & 0 & -0.02 \\ 0 & -3 & 0 \\ 0.98 & 0 & 0.006 \end{bmatrix}, \quad B_{q_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_{q_1} = \begin{bmatrix} 0.95 \\ 0 \\ -0.31 \end{bmatrix}$$

$$A_{q_2} = \begin{bmatrix} -4 & 0 & -0.02 \\ 0 & -2 & 0 \\ 0.98 & 0 & -0.99 \end{bmatrix}, \quad B_{q_2} = \begin{bmatrix} 0.31 \\ 0 \\ 0.95 \end{bmatrix}, \quad C_{q_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T$$

The system above is minimal, but none of the subsystems is minimal
Example: cont

If we simulate the two systems for white noise input and switching sequence \((q_2, 1)(q_1, 2)(q_1, 3)(q_2, 1)\).
Further consequences

- For linear switched systems which are observable and span-reachable from zero, the input-output function from 0 captures all the input-output behavior.

- It is impossible to estimate the state for non-observable linear switched systems. The converse need not be true.

- It is impossible to control (stabilize) a linear switched system with continuous inputs, if it is not span-reachable from zero. The converse need not be true.

- Minimal switched systems isomorphic $\implies$ control depends only on the input-output behavior not on the choice of the state-space representation.

- Existence of quadratic (control) Lyapunov functions, storage functions is a property of input-output behavior.
Linear Time Invariant (LTI) state-space representation

\[ \Sigma = (A, B, C). \]

Input-output map \( Y_\Sigma = l_{A,B,C,0} \) maps input \( u(.) \) to output \( y(.) \), initial state \( x(0) = 0 \).

\[
Y_\Sigma(u)(t) = \begin{cases} 
\int_0^t Ce^{A(t-s)} Bu(s) ds \\
\sum_{s=0}^{t-1} CA^{(t-s)} Bu(s)
\end{cases}
\]

\( \Sigma \) is a realization of \( Y : u(.) \mapsto y(.) \), iff \( Y_\Sigma = Y \).

Realization problem

For the specified input-output map \( Y \) find a (preferably minimal) linear system \( \Sigma \) such that \( \Sigma \) realizes \( Y \).
Impulse response

A potential input-output map of a linear system is determined by its impulse response:

**Impulse response** \( G(t) \)

\[
Y(u(\cdot), t) = \begin{cases} 
\int_0^t G(t - s)u(s)ds & \text{continuous time} \\
\sum_{s=0}^{t-1} G(t - s)u(s) & \text{discrete time}
\end{cases}
\]

\( \Sigma \) is a realization, iff

\[
G(t) = Ce^{At}B \quad \text{(cont.time)}
\]
\[
G(t) = CA^tB \quad \text{(disc.-time)}
\]
Markov parameters

\[ M_k = \begin{cases} \frac{d^k}{dt^k} G(t)|_{t=0} & \text{continuous time, or} \\ G(k + 1) & \text{discrete time} \end{cases} \]

Classical step. \( \Sigma \) is a realization of \( Y \iff M_k = CA^kB \)
Existence of a realization

Recall $M_k$ – Markov parameters*

Hankel matrix of $Y$ \[ H_Y = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots \\ M_1 & M_2 & M_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \]

Theorem

- $Y$ has a realization by an LTI $\iff \text{rank } H_Y < +\infty$.
- rank $H_Y$ is the dimension of a minimal LTI realization of $Y$. 

*Note: $M_k$ refers to Markov parameters in the context of system theory.
Ho-Kalman algorithm

1. Find a factorization

\[
H_{N,N+1} = \begin{bmatrix}
    M_0 & M_1 & \cdots & M_N \\
    M_1 & M_2 & \cdots & M_{N+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    M_{N-1} & M_N & \cdots & M_{2N-1}
\end{bmatrix} = OR
\]

s.t. \( O \) full column rank, \( R \) full row rank.
(e.g, SVD: \( H_{N,N+1} = U\Sigma V^T \), \( O = U\Sigma^{1/2} \), \( R = \Sigma^{1/2} V^T \)).

2. \( R = [R_1, R_2, \cdots, R_{N+1}] \), \( O = \begin{bmatrix}
    O_1 \\
    O_2 \\
    \vdots \\
    O_N
\end{bmatrix} \).

3. \( B = R_1 \), \( C = O_1 \), and \( A \) solves

\[
A [R_1, R_2, \cdots, R_N] = [R_2, R_3, \cdots, R_{N+1}]
\]
Correctness of Ho-Kalman algorithm and partial realization

\[ H_{N,N} = \begin{bmatrix} M_0 & M_1 & \cdots & M_{N-1} \\ M_1 & M_2 & \cdots & M_N \\ \vdots & \vdots & \ddots & \vdots \\ M_{N-1} & M_N & \cdots & M_{2N-2} \end{bmatrix}, \]

\[ H_{N+1,N} = \begin{bmatrix} M_0 & M_1 & \cdots & M_{N-1} \\ M_1 & M_2 & \cdots & M_N \\ \vdots & \vdots & \ddots & \vdots \\ M_N & M_{N+1} & \cdots & M_{2N-1} \end{bmatrix} \]
Correctness of Ho-Kalman algorithm and partial realization

**Theorem (Ho-Kalman algorithm & partial realization)**

- \( \text{rank } H_{N,N} = \text{rank } H_Y \implies (A, B, C) \) is a minimal realization of \( Y \)

- If \( Y \) has a realization of dimension less than \( N \), then \( \text{rank } H_{N,N} = \text{rank } H_Y \).

- \( \text{rank } H_{N,N} = \text{rank } H_{N+1,N} = \text{rank } H_{N,N+1} \implies (A, B, C) \) is a so called \( 2N \) realization of \( Y \), i.e.

\[
M_k = CA^kB, \quad k = 0, 1, \ldots, 2N - 1
\]
Impulse response of linear switched systems

- Potential input-output map $Y$ of a linear switched system
  1. Maps switching signal $q(.)$ and input $u(.)$ to output $y(.)$.
  2. Linear in continuous input $u(.)$.

- $Y$ is completely described by its impulse response

**Impulse response for switching $q(.)$**

Switching $q()$: stay in discrete mode $q_1, \ldots, q_k$ for times $t_1, \ldots, t_k$.

$$G_{q_1 \ldots q_k}(t_1, \ldots, t_k) = Y(q(.), \sigma_0)$$

- $\sigma_0$ is the Dirac-delta for continuous-time
- $\sigma_0(0) = 1$, $\sigma_0(t) = 0$, $t > 0$ for discrete-time
Markov parameters, $q_0, q \in Q$ – discrete modes, $j = 1, 2, \ldots, m$

$$S_{q, q_0}(q_1 q_2 \cdots q_k) = \begin{cases} G_{q_0 q_1 \cdots q_k q}(1, 1, \ldots, 1) \\ \frac{d}{dt_1} \cdots \frac{d}{dt_k} G_{q_0 q_1 \cdots q_k q}(0, t_1, \ldots, t_k, 0) \bigg|_{t_1=\cdots=t_k=0} \end{cases}$$

Markov parameters are indexed by sequences of discrete modes $Q^*$

$\Sigma$ is a realization of $Y \iff$

$$S_{q, q_0}(q_1 q_2 \cdots q_k) = C_q A_{q_k} \cdots A_{q_1} B_{q_0}$$
Hankel matrix for linear switched systems

\[ Q = \{1, 2, \ldots, D\} \]

\[ v_1 \prec \ldots \prec v_k, \ldots \] lexicographic ordering of all sequences.

\[
M(v) = \begin{bmatrix}
S_{1,1}(v) & \ldots & S_{1,D}(v) \\
\vdots & \ldots & \vdots \\
S_{D,1}(v) & \ldots & S_{D,D}(v)
\end{bmatrix}
\]

Hankel matrix: \( H_Y \)

\[
H_Y = \begin{bmatrix}
M(v_1 v_1) & M(v_2 v_1) & \ldots & M(v_k v_1) & \ldots \\
M(v_1 v_2) & M(v_2 v_2) & \ldots & M(v_k v_2) & \ldots \\
M(v_1 v_3) & M(v_2 v_3) & \ldots & M(v_k v_3) & \ldots \\
\vdots & \vdots & \ldots & \vdots & \ldots
\end{bmatrix}
\]
Realization theorem for linear switched systems

Theorem (Pet06, Pet07, Pet11a, Pet13)

- $Y$ has a realization $\iff \text{rank } H_Y < +\infty$, $+$.
Realization algorithm [Pet06, Pet11, Pet13]

\[
H_{Y,N+1,N} = \begin{bmatrix}
M(v_1 v_1) & \cdots & M(v_{M(N)} v_1) \\
\vdots & \ddots & \vdots \\
M(v_1 v_{M(N)}) & \cdots & M(v_{M(N)} v_{M(N)}) \\
M(v_1 v_{M(N+1)}) & \cdots & M(v_{M(N)} v_{M(N+1)})
\end{bmatrix}
\]

**M(N)** – number of sequences over Q of length at most N

1: \(H_{f,N+1,N} = OR\)
2: \(B_q = m(q - 1) + 1, \ldots, mq\)th columns of \(R\).
3: \(C_q = p(q - 1) + 1, \ldots, pq\)th rows of \(O\).
4: \(A_q = \bar{O}^+ O_q\)

- \(\bar{O}\) – the block rows of \(O\) which are indexed by \(v_1, \ldots, v_N\).
- \(\bar{O}^+\) – pseudo-inverse of \(\bar{O}\).
- \(O_q\) – shifted \(\bar{O}\): the row of \(O_q\) indexed by sequence \(v\) is the row of \(O\) indexed by sequence \(qv\).
Partial realization theorem for linear switched systems

\[ H_{Y,N,N} = \begin{bmatrix}
    M(v_1 v_1) & \cdots & M(v_{M(N)} v_1) \\
    \vdots & \ddots & \vdots \\
    M(v_1 v_{M(N)}) & \cdots & M(v_{M(N)} v_{M(N)})
\end{bmatrix} \]

\[ H_{Y,N,N+1} = \begin{bmatrix}
    M(v_1 v_1) & \cdots & M(v_{M(N)} v_1) & M(v_{M(N+1)} v_1) \\
    \vdots & \ddots & \vdots & \vdots \\
    M(v_1 v_{M(N)}) & \cdots & M(v_{M(N)} v_{M(N)}) & M(v_{M(N+1)} v_{M(N)})
\end{bmatrix} \]

Theorem (Pet11b,Pet13)

1. If \( \text{rank } H_{Y,N,N} = \text{rank } H_{Y,N,N+1} = \text{rank } H_{Y,N+1,N} \) then the result of the algorithm recreates the Markov-parameters \( M(v_1), \ldots, M(v_{M(2N+1)}) \).

2. If \( N \geq \) the dimension of a realization of \( Y \), then the algorithm returns a minimal realization of \( Y \).
Example

Consider the switched system from the previous example and let $Y$ be the input-output map of that system.

\[ H_{Y,2,1} = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & -1 \\
1 & 0 & -3 & 0 & -2 & 0 \\
0 & -1 & 0 & 3 & 0 & 4 \\
-3 & 0 & 9 & 0 & 6 & 0 \\
0 & -1 & 0 & 4 & 0 & 5 \\
-2 & 0 & 6 & 0 & 4 & 0 \\
0 & 3 & 0 & -9 & 0 & -12 \\
9 & 0 & -27 & 0 & -18 & 0 \\
0 & 4 & 0 & -12 & 0 & -16 \\
6 & 0 & -18 & 0 & -12 & 0 \\
0 & 4 & 0 & -12 & 0 & -16 \\
6 & 0 & -18 & 0 & -12 & 0 \\
0 & 5 & 0 & -16 & 0 & -21 \\
4 & 0 & -12 & 0 & -8 & 0
\]
Example: cont.

Applying the realization algorithm to $H_{Y,2,1}$ yields.

$$A_{q_1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3.02 & 0.17 \\ 0 & -0.32 & 0.018 \end{bmatrix}, \quad B_{q_1} = \begin{bmatrix} -1.9 \\ 0 \\ 0 \end{bmatrix}, \quad C_{q_1} = \begin{bmatrix} 0 \\ 0.21 \\ 0.46 \end{bmatrix}$$

$$A_{q_2} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4.02 & 0.17 \\ 0 & -0.32 & -0.98 \end{bmatrix}, \quad B_{q_2} = \begin{bmatrix} 0 \\ 1.25 \\ -0.57 \end{bmatrix}, \quad C_{q_2} = \begin{bmatrix} -0.53 \\ 0 \\ 0 \end{bmatrix}$$
Example: cont

If we simulate the two systems for white noise input and switching sequence \((q_2, 1)(q_1, 2)(q_1, 3)(q_2, 1)\).
Further work

- The results above can be extended to linear jumps and bilinear local equations.
- The results can be extended to LPV systems.
- Extension to stochastic jump-Markov linear systems.
- Application to model reduction, system identification.
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