Robust stability analysis of discrete-time systems
with parametric and switching uncertainties

Dimitri Peaucelle

Yoshio Ebihara

To be presented at 19th IFAC World Congress / Cape Town
Motivation

- Analysis of discrete-time polytopic systems $x_{k+1} = A(\theta_k)x_k$ where
  
  $A(\theta_k) = \sum_{v=1}^{\bar{v}} \theta_{k,v} A^{[v]} : \theta_k \in \Xi_{\bar{v}} = \left\{ \sum_{v=1}^{\bar{v}} \theta_{k,v} = 1, \theta_{k,v} \geq 0 \right\}$

1. “Quadratic stability” test [Bar85] $\theta_{k+1} \neq \theta_k$

   $\exists P \succ 0 : A^{[v]T}PA^{[v]} - P \prec 0 \ \forall v = 1\ldots\bar{v}$

2. “Slack variables” test [PABB00] assuming $\theta_k = \theta \ \forall k \in \mathbb{N}$

   $\exists P^{[v]} \succ 0, G : \begin{bmatrix} P^{[v]} & 0 \\ 0 & -P^{[v]} \end{bmatrix} \prec \left\{ G \left[ \begin{array}{c} I \\ -A^{[v]} \end{array} \right] \right\}^S \ \forall v = 1\ldots\bar{v}$

3. “Switching” test [DB01] $\theta_{k+1} \neq \theta_k$

   $\exists P^{[v]} \succ 0 : A^{[v]T}P^{[w]}A^{[v]} - P^{[v]} \prec 0 \ \forall v = 1\ldots\bar{v} \ \forall w = 1\ldots\bar{v}$

- $2 \iff 1$ and $3 \iff 1$ but $2 \not\iff 3$
Descriptor uncertain models

- Any rationally dependent model admits an LFT representation

\[
x_{k+1} = A(\theta_k)x_k : A(\theta) = A + B\Delta(\theta)(I - D\Delta(\theta))^{-1}C
\]

where \(\Delta(\theta)\) is linear in \(\theta\).

- Also admits a reduced size descriptor representation

\[
E_x(\theta_k)x_{k+1} + E_\pi(\theta_k)\pi_k = F(\theta_k)x_k
\]

where \(E_x(\theta), E_\pi(\theta), F(\theta)\) are affine in \(\theta\)

- Proof: take

\[
\begin{bmatrix}
I \\
0
\end{bmatrix} x_{k+1} + \begin{bmatrix}
-B\Delta(\theta_k) \\
I - D\Delta(\theta_k)
\end{bmatrix} \pi_k = \begin{bmatrix}
A \\
C
\end{bmatrix} x_k
\]

- Remark: \(\begin{bmatrix}
E_x(\theta) & E_\pi(\theta)
\end{bmatrix}\) is full column rank if LFT is well-posed.
Example: \( a_k y_{k+2} + b_k^2 y_{k+1} + a_k b_k y_k = 0 \)

**LFT model:** \( \Delta(\theta) \) is at least \( 4 \times 4 \) (i.e. model with 8 exogenous signals), e.g.

\[
A(a,b) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Delta(I - \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix})^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

with \( \Delta = \begin{bmatrix} bI_2 & 0 \\ 0 & (a - 1)I_2 \end{bmatrix} \)

**Descriptor model with 1 exogenous signal**

\[
\begin{bmatrix} a_k & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x_{k+1} + \begin{bmatrix} b_k \\ 0 \\ 1 \end{bmatrix} \pi_k = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ b_k & a_k \end{bmatrix} x_k
\]
Robust switching SV test for descriptor systems

- Polytopic descriptor system with uncertain and witching parameters:

\[ E_x(\phi, \theta_k)x_{k+1} + E_\pi(\phi, \theta_k)\pi_k - F(\phi, \theta_k)x_k = M(\phi, \theta_k) \begin{pmatrix} x_{k+1} \\ \pi_k \\ x_k \end{pmatrix} = 0 \]

\[ M(\phi, \theta_k) = \sum_{\mu=1}^{\bar{\mu}} \sum_{v=1}^{\bar{v}} \phi_{\mu} \theta_{k,v} M^{[\mu,v]} : \phi \in \Xi_{\bar{\mu}}, \theta_k \in \Xi_{\bar{v}} \]

△ System assumed well-posed: \( \begin{bmatrix} E_x(\phi, \theta) & E_\pi(\phi, \theta) \end{bmatrix} \) is strict full column rank.

■ Stability is assed if \( \exists P^{[\mu,v]} \succ 0, G^{[v]} : \)

\[
\begin{bmatrix}
P^{[\mu,w]} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -P^{[\mu,v]}
\end{bmatrix}
\prec \{ G^{[w]} M^{[\mu,v]} \}_S
\]

\( \forall \mu = 1 \ldots \bar{\mu} \)

\( \forall v = 1 \ldots \bar{v} \)

\( \forall w = 1 \ldots \bar{v} \)

★ Contains all previous results as special cases. High numerical complexity.
Robust switching SV test for descriptor systems

\[
\begin{bmatrix}
P[\mu,w] & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -P[\mu,v]
\end{bmatrix}
\prec \begin{bmatrix}
G[w]M[\mu,v]
\end{bmatrix}^S
\forall \mu = 1 \ldots \bar{\mu}
\forall v = 1 \ldots \bar{v}
\forall w = 1 \ldots \bar{v}
\]

- Numerical complexity can be reduced if
  - some of the columns of \( E_x(\phi, \theta) \quad E_\pi(\phi, \theta) \) are independent of \( \phi, \theta \)
  - or some of the rows of \( M(\phi, \theta) \) are independent of \( \phi, \theta \).

- Example of the switching ordinary system \( M(\theta) = \begin{bmatrix} I & -A(\theta) \end{bmatrix} \)
  - lossless reduced size LMI (no need for slack variables)

\[
A[v]^T P[w] A[v] - P[v] \prec 0 \quad \forall v = 1 \ldots \bar{v} \quad \forall w = 1 \ldots \bar{v}
\]
Robust switching SV test for descriptor systems

- Large size LMI for the numerical example

\[
P(P^{[\mu,w]}, P^{[\mu,v]}) \preceq \begin{cases} 
G^{[w]} & \\
\begin{bmatrix} 
a[\bullet] & b[\bullet] & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & -b[\bullet] & -a[\bullet]
\end{bmatrix}
\end{cases}
\]

- Lossless reduced size LMIs (both in nb of rows and nb of vars)

\[
\hat{P}(P^{[\mu,w]}, P^{[\mu,v]}) \preceq \begin{cases} 
\hat{G}^{[w]} & \\
\begin{bmatrix} 
a[\bullet] & 0 & b[\bullet] & 0 \\
0 & -b[\bullet] & 1 & -a[\bullet]
\end{bmatrix}
\end{cases}
\]

- $\bullet = \mu$ if $a$ is parametric (constant) and $\bullet = v$ if $a_k$ is switching
- $\star = \mu$ if $b$ is parametric (constant) and $\star = v$ if $b_k$ is switching
Robust switching SV test for descriptor systems

Numerical results for $a \in [1, 2]$ and $b \in [-0.5, \beta]$.

Goal: find maximal $\beta$ such that LMIs are feasible

<table>
<thead>
<tr>
<th>$a_k, b_k$</th>
<th>$\beta$ (nb vars/nb rows)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a, b_k$</td>
<td>0.81094 (44/64)</td>
</tr>
<tr>
<td>$a_k, b$</td>
<td>0.89027 (28/32)</td>
</tr>
<tr>
<td>$a, b$</td>
<td>0.82658 (28/32)</td>
</tr>
<tr>
<td></td>
<td>0.98059 (20/16)</td>
</tr>
</tbody>
</table>
Conservatism reduction by system augmentation

Considered example, model before augmentation:

\[ a_k y_{k+2} + b_k^2 y_{k+1} + a_k b_k y_k = 0 \]

After one step ahead augmentation:

\[
\begin{align*}
    a_k y_{k+2} + b_k^2 y_{k+1} + a_k b_k y_k &= 0 \\
    a_{k+1} y_{k+3} + b_{k+1}^2 y_{k+2} + a_{k+1} b_{k+1} y_{k+1} &= 0
\end{align*}
\]

Corresponding descriptor representation

\[
\begin{bmatrix}
    a_{k+1} & 0 & 0 \\
    0 & a_k & 0 \\
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{bmatrix} \tilde{x}_{k+1} + \begin{bmatrix}
    b_{k+1} & 0 \\
    0 & b_k \\
    0 & 0 \\
    0 & 0 \\
    1 & 0 \\
    0 & 1
\end{bmatrix} \tilde{\pi}_k = \begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    1 & 0 & 0 \\
    0 & 1 & 0 \\
    b_{k+1} & a_{k+1} & 0 \\
    0 & b_k & a_k
\end{bmatrix} \tilde{x}_k
\]

Results applied to augmented model are guaranteed to be less conservative
Conservatism reduction by system augmentation

- **Numerical results for** $a \in [1, 2]$ and $b \in [-0.5, \beta]$.
- **Goal:** find maximal $\beta$ such that LMIs are feasible

<table>
<thead>
<tr>
<th>$\beta$ (nb vars/nb rows)</th>
<th>original model</th>
<th>augmented</th>
<th>upper limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_k, b_k$</td>
<td>0.81094 (44/64)</td>
<td>0.84677 (480/1536)</td>
<td>?</td>
</tr>
<tr>
<td>$a, b_k$</td>
<td>0.89027 (28/32)</td>
<td>0.90293 (144/192)</td>
<td>?</td>
</tr>
<tr>
<td>$a_k, b$</td>
<td>0.82658 (28/32)</td>
<td>0.85375 (144/192)</td>
<td>?</td>
</tr>
<tr>
<td>$a, b$</td>
<td>0.98059 (20/16)</td>
<td>0.99519 (48/24)</td>
<td>1</td>
</tr>
</tbody>
</table>
Conclusions

- New LMI test
  - Combines existing techniques w.r.t. uncertain & switching parameters
  - Extends results to descriptor-like models (represent any rationally-dependent system)
  - Contributes to conservatism reduction thanks to the system augmentation technique
  - Numerical complexity is increased ... but can be controlled

Prospective work
- Continuous time case
- Time-varying parameters with bounded rate
- Extension to design problems (state & output feedback, robust observers...)

Springer monograph by Y. Ebihara & D. Peaucelle to be published in 2014-2015
References

