A switched LQ regulator design in continuous time

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Plan

Problem formulation and preliminary results

Numerical resolution

Lyapunov based switching law

Discussion concerning the switching law and its degree of optimality

Illustrative examples
   Example 1: Stabilizable subsystems
   Example 2: Non stabilizable subsystems

Conclusion
Problem formulation and preliminary results

Consider the class of linear switched systems in continuous time:

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t) \quad x(0) = x_0
\]  

(1)

where

- $\sigma : [0, +\infty) \rightarrow S = \{1, \cdots, s\}$.
- $(A_i, B_i) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m_i}, i \in S$,
- $u_i(t) \in \mathbb{R}^{m_i}, 0 \leq m_i \leq n$

Objective: Design a state feedback switching law (i.e. $x \mapsto (\sigma(x), u_{\sigma(x)}(x))$) that approaches the optimal solution of the following optimization problem:

Problem 1: Minimize the switched quadratic criterion:

\[
\min_{\sigma, u_\sigma} \frac{1}{2} \int_0^\infty x^T Q_{\sigma} x + u_{\sigma}^T R_{\sigma} u_{\sigma} \, dt
\]  

(2)

where $Q_i = Q_i^T > 0, R_i = R_i^T > 0, i \in S$

Up to now the exact solution is not available and only approximation via dynamic programming and (open loop) numerical solutions are available.
Problem formulation and preliminary results

**Framework:** Reformulate Problem 1 into

**Problem 2:** Minimize the quadratic criterion:

\[
\min_{\lambda(t), u_i(t)} \frac{1}{2} \int_0^\infty \sum_{i=1}^s \lambda_i (x^T Q_i x + u_i^T R_i u_i) dt
\]

subject to

\[
\dot{x} = \sum_{i=1}^s \lambda_i (A_i x + B_i u_i), \quad x(0) = x_0, \quad \lambda(t) \in \Lambda = \left\{ \lambda \in \mathbb{R}^s : \sum_{i=1}^s \lambda_i = 1, \lambda_i \geq 0 \right\}.
\]

Three reasons justify the convexification of the problem:

1. The solutions are well defined [Filippov, 1988]
2. The density of the switched system trajectories into the trajectories of its relaxed version [Ingalls - Sontag 2002]
3. The existence of *singular* optimal solutions are taking into account [Patino-Riedinger 2009, Bengea-Decarlo 2005].
To apply Pontryagin Maximum Principle (PMP) for Problem 1 or its relaxed version, the Hamiltonian function is defined as follow:

$$\mathcal{H}(x, \lambda, u, p) = \sum_{i=1}^{s} \lambda_i \mathcal{H}_i(x, u_i, p)$$

(3)

with $$\mathcal{H}_i(x, u_i, p) = p^T(A_i x + B_i u_i) + \frac{1}{2}(x^T Q_i x + u_i^T R_i u_i)$$ and where $$p$$ defines the co-state.

**Theorem (1)**

Suppose that $$(\lambda^*, u^*)$$ is optimal with the corresponding state $$x^*$$. Then, there exists an absolutely continuous function $$p^*$$, named co-state, such that:

1. $$p^* \neq 0,$$
2. $$\dot{p}^* = \sum_{i=1}^{s} \lambda_i^*(t)(-A_i^T p^* - Q_i x^*)$$ for almost all $$t \in \mathbb{R}^+,$$
3. $$(\lambda^*(t), u^*(t)) \in \arg \min_{(\lambda \in \Lambda, u)} \mathcal{H}(x^*(t), \lambda, u, p^*(t)),$$
4. $$\mathcal{H}(x^*(t), \lambda^*(t), u^*, p^*(t)) = 0.$$
Problem formulation and preliminary results

As the minimum of $\mathcal{H}$ with respect to the $u_i$’s is clearly independent of the value of $\lambda$, Theorem 1 can be simplified:

**Lemma**
The optimal value of the $u_i$’s are given by $u_i^*(t) = -R_i^{-1}B_i^T p^*(t)$ and $\lambda^*$ satisfies:

$$
\lambda^*(t) \in \arg \min_{\lambda \in \Lambda} \sum_{i=1}^{s} \lambda_i \mathcal{H}_i(x^*, -R_i^{-1}B_i^T p^*, p^*). 
$$

Thus, optimal controls $\lambda^*$ satisfy the complementarity constraints:

$$
0 \leq \lambda_i^* \perp \mathcal{H}_i(x^*, -R_i^{-1}B_i^T p^*, p^*) \geq 0, \ i \in S
$$

the sign $x \perp y$ means $xy = 0$. 

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The symbol $\perp$ in the context of complementarity constraints denotes orthogonality in the context of linear algebra, indicating that $x$ and $y$ are orthogonal if their dot product is zero.
Numerical resolution

Major drawback in the numerical resolution: The existence of singular controls.

Singular controls: there exist at least two indices \((i, j) \in S^2\) such that on a non-empty time interval \((a, b)\),

\[ H_i = H_j = 0, \forall t \in (a, b) \]

Then all values satisfying \(\lambda_i + \lambda_j = 1\) are potential candidate for optimality

- PMP is inconclusive concerning the value of \(\lambda^*\) (Additional NC are required)
- \(\lambda\) is not admissible for the switched systems (not at the vertices of \(\Lambda\)) but could be approximated by chattering (Thanks to density theorem).

Numerical consequences:
- Indirect methods like shooting methods are inoperative
  - the uniqueness of the solution of Hamiltonian system is lost (bifurcations)
  - the solution structure (regular -singular) is required
- Direct methods (NLP) yield to bad numerical results due to the insensitivity of the Lagrangian w.r.t. the control
Numerical resolution

Idea: Take implicitly into account the singular arcs using the necessary condition of the PMP and the Hamiltonian systems and then solve directly an augmented constraint optimization problem.

Denote by $z = (x, p)$

**Problem 2:** Minimize (using NLP):

$$
\min_{\lambda(t)} \frac{1}{2} \int_{0}^{\infty} \sum_{i=1}^{s} \lambda_i (x^T Q_i x + p^T B_i R^{-1}_i B_i^T p) dt
$$

subject to

$$
\dot{z} = \sum_{i=1}^{s} \lambda_i \begin{pmatrix}
A_i & -B_i R^{-1}_i B_i^T \\
-Q_i & -A_i^T
\end{pmatrix} z
$$

$$
0 \leq \lambda_i \perp \mathcal{H}_i(x, -R^{-1}_i B_i^T p, p) \geq 0, \quad i \in S
$$

$$
\lambda(t) \in \Lambda, \quad x(0) = x_0
$$

where the sign $x \perp y$ means $xy = 0$.

**Special issue:** Discontinuous Differential Systems: Theory and Numerical Methods

P. Riedinger, C. Morarescu, A numerical framework for optimal control of switched input affine nonlinear systems subject to path constraint, Mathematics and Computers in Simulation, January 2014
Lyapunov based switching law

**Insight:** Lyapunov function as a tight upper bound on the value function (may coincide at some points)

- Consider the family of Riccati equations parametrized by $\lambda \in \Lambda$:

$$
A(\lambda)^T P_\lambda + P_\lambda A(\lambda) - P_\lambda B(\sqrt{\lambda}) R^{-1} B(\sqrt{\lambda})^T P_\lambda + Q(\lambda) = 0.
$$

(8)

- corresponding to the LQ subproblem obtained for a fixed $\lambda$, if exists.

- $A(\lambda) = \sum_{i \in S} \lambda_i A_i$,
- $B(\sqrt{\lambda}) = [\sqrt{\lambda_1} B_1 | \sqrt{\lambda_2} B_2 | \ldots | \sqrt{\lambda_s} B_s ]$
- $Q(\lambda) = \sum_{i \in S} \lambda_i Q_i$ and $R = \text{diag}(\{R_1, R_2, \ldots, R_s\})$.

**Lemma**

*If the pair $(A(\lambda), B(\sqrt{\lambda}))$ is stabilizable and $Q(\lambda)$ is positive definite, then there exists a positive definite solution to the parametrized Riccati equation Eq. (8).*
Lyapunov based switching law

We denote by $\Lambda^+$ the set

$$\Lambda^+ = \{ \lambda \in \Lambda \mid \text{the pair } (A(\lambda), B(\sqrt{\lambda})) \text{ is stabilizable and } \max \text{spec}(P_\lambda) \leq \nu_{\max}\}$$

where $\text{spec}(P_\lambda)$ denotes the spectrum of $P_\lambda$ and $\nu_{\max}$ an arbitrary large number.

$\Lambda^+$ satisfies the following property.

**Lemma**

The matrices $Q_i$ being positive definite, if one can find $\lambda^0 \in \Lambda$ such that $(A(\lambda^0), B(\sqrt{\lambda^0}))$ is controllable, then, for every $\nu_{\max}$ large enough, set $\Lambda^+$ is compact and its interior is not empty in $\Lambda$.

Moreover, the two following real numbers, $\alpha_m$ and $\alpha_M$, defined as

$$\alpha_m = \min_{\lambda \in \Lambda^+} \min (\text{spec}(P_\lambda)) \quad \alpha_M = \max_{\lambda \in \Lambda^+} \max (\text{spec}(P_\lambda))$$

are positive.
Lyapunov based switching law

Let us now introduce the following Lyapunov function

$$V_m(x) := \inf_{\lambda \in \Lambda^+} x^T P_\lambda x$$

(9)

where $P_\lambda$ denotes the solution of Riccati equation (8).

- We show that $V_m$ is a positive definite function, homogeneous of degree 2, proper and locally Lipschitz.
- Moreover, the directional derivative of $V_m(x; d)$ in direction $d$ is given by [Furukawa 1983]:

$$V'_m(x; d) = \lim_{h \to 0; h > 0} \frac{V_m(x + hd) - V_m(x)}{h} = 2 \inf_{\lambda \in \ell(x)} d^T P_\lambda x.$$  

where $\ell(x)$ denotes the subset of $\lambda \in \Lambda^+$ such that $V_m(x) = x^T P_\lambda x$. 
Lyapunov based switching law

Theorem (Main result)

Assume that

1. \( Q_i > 0, i \in S \)
2. \( \exists \lambda_0 \text{ s.t. } (A(\lambda_0), B(\sqrt{\lambda_0})) \text{ is controllable.} \)

For every \( x \in \mathbb{R}^n \), we choose

\[
(i(x), \lambda(x)) \in \arg \min_{(i, \lambda) \in S \times \ell(x)} \left( 2x^T M_i(\lambda) P_{\lambda} x + x^T N_i(\lambda) x \right).
\]

where

\[
M_i(\lambda) := A_i - B_i K_i(\lambda),
\]
\[
K_i(\lambda) := R_i^{-1} B_i^T P_{\lambda} \]
\[
N_i(\lambda) := Q_i + K_i(\lambda)^T R_i K_i(\lambda).
\]

Then, the feedback

\[
\sigma = i(x),
\]
\[
u_i(x) = -K_i(x)(\lambda(x))x = -R_i^{-1} B_i^T P_{\lambda(x)} x
\]

stabilizes the switched system (1) with a cost smaller than \( \frac{1}{2} V_m(x_0) \).

Exponential convergence rate is greater than \( \beta = \frac{\eta_0}{\alpha_1} \) where \( \eta_0 \) and \( \alpha_1 \) are given by:

\[
\eta_0 = \min_{i \in S} \inf_{x \in S^{n-1}} \inf_{\lambda \in \ell(x)} x^T N_i(\lambda) x, \quad \alpha_1 = \max_{x \in S^{n-1}} V_m(x)
\]
Lyapunov based switching law

Sketch of the proof:

- Riccati eq. (8) can be rewritten as a convex combination:
  \[ \sum_{i \in S} \lambda_i (2x^T M_i^T(\lambda) P_{\lambda} x + x^T N_i(\lambda) x) = 0, \]

- For every \((x, \lambda) \in \mathbb{R}^n \times \Lambda^+\),
  \[ \min_{i \in S} \left( 2x^T M_i^T(\lambda) P_{\lambda} x + x^T N_i(\lambda) x \right) \leq 0 \]

- Then, from the directional derivative of \(V_m\), for every \((x, \lambda^0) \in \mathbb{R}^n \times \ell(x)\), there exists \(i(x, \lambda^0)\) such that in direction \(d = M_{i(x, \lambda^0)}(\lambda^0) x\)
  \[ V_m'(x; M_i(\lambda^0) x) \leq 2x^T M_i^T(\lambda^0) P_{\lambda^0} x \leq -x^T N_i(\lambda^0) x \]

- Therefore, for any initial condition \(x_0\),
  \[ V_m(x(t)) + \int_0^t x^T (Q_i(x) + K_i(x(\lambda(x)))^T R_i(x) K_i(x(\lambda(x))) x d\tau \leq V_m(x_0), \quad \forall t \geq 0. \]

As \(Q_i > 0, \forall i \in S\), it follows that: \(x(t) \to 0\) when \(t \to +\infty\).
Discussion concerning the switching law and its optimality

Why do we claim that the Lyapunov function can be a tight upper bound on the value function?

- The value \( \frac{1}{2} V_m(x) \) is the best cost related to every constant convex combination that stabilizes the relaxed system (In infinite number!).
- If all subsystems are stabilizable, then \( \frac{1}{2} V_m(x) \leq \min_{i \in S} \frac{1}{2} x^T P_i x \)

When \( \frac{1}{2} V_m(x) \) is optimal?
“Along the part of trajectories where the optimal control \( \lambda^* \) is constant to reach the origin”.

- if the number of switchings is finite
- if the trajectory is steered to the origin by a constant singular control \( \lambda \) for which \( P_\lambda > 0 \).
  → Singular controls in dimension \( n = 2 \) are constant.
Formally, we can justify the design of the switching law as follows.

- Assuming known the value function, one can write for any $T > 0$, 
\[
V^*(x_0) = \min_{\sigma} \frac{1}{2} \int_0^T x^T Q_{\sigma(t)} x + u_{\sigma(t)}^T R_{\sigma(t)} u_{\sigma(t)} \, dt + V^*(x(T))
\]
- The transversality condition of PMP implies at time $T$, $p^*(T) = \frac{\partial V(x(T))}{\partial x}$ (if it exists).
- Now suppose that $V^*(x(T))$ is approximated by $V_m(x(T))$. Then, an approximation of $p^*(T)$ is given by $p^*(x(T)) \approx P_{\lambda} x(T)$ with $\lambda \in \ell(x)$.
- Thus, it is easy to check that the minimization of the Hamiltonian at time $T$ leads to the provided switching law.
- As the problem is homogenous and if the approximation is "good", one can infer that $p^*(x) \approx P_{\lambda(x)} x$ with $\lambda(x) \in \ell(x)$ for every $x$.

Roughly speaking, the state feedback switching law matches the optimal one when $P_{\lambda(x)} x$ is a good approximation of $p^*$.
Example 1

Consider a two mode switched system with the following design parameters:

\[ A_1 = \begin{pmatrix} -2.7 & 3.9 \\ 4.4 & -12.6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -9.5 & -5.1 \\ -7.5 & -3.3 \end{pmatrix}, \]

\[ B_1 = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 4.6 \\ 0 \end{pmatrix}, \]

\[ Q_1 = Q_2 = \text{Id}, \ R_1 = 1 \text{ and } R_2 = 2. \]

For each subsystem, an LQ design can be performed separately.
Example 1:

Figure: Ex. 1: State space trajectories: (red) optimal solution (NLP); (blue) switching law

Figure: Ex. 1: Cost comparisons for different initial positions taken on the unit ball.
Example 2

For this second example, we have chosen two non stabilizable subsystems:

\[ A_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -2 & 1 \\ 0 & 1 \end{pmatrix}, \]

\[ B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \]

\[ Q_1 = Q_2 = \text{Id}, \quad R_1 = 2 \text{ and } R_2 = 1 \]

- There is no LQ design that can be defined separately for each subsystem.
- The set \( \Lambda^+ \) is non empty, the switching law presented in this paper can be applied.
Example 2:

Figure: Ex. 2: State space trajectories: (red) optimal solution (NLP); (blue) switching law

Figure: Ex. 2: Cost comparisons for different initial positions taken on the unit ball.
A state feedback switching law for switched LQ regulator problems in continuous time.

Applicable if a controllable convex combination of the subsystems exists.

The switching law can be optimal along arcs (singular or not) ending to the origin with a constant optimal control.

In any case, a guarantee on the cost is provided by the upper bound $\frac{1}{2} V_{min}(x)$.

Additional stability results in the paper for sampled switched law.

Related papers:

- P. Riedinger, A switched LQ regulator design in continuous time, IEEE TAC to appear in May 2014.