Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive systems with applications to asynchronous sampled-data systems

Corentin Briat
Introduction
Impulsive systems

Linear case

\[\begin{align*}
\dot{x}(t) &= Ax(t), \quad t \notin \{t_k\}_{k \in \mathbb{N}_0} \\
x(t) &= Jx(t^-), \quad t \in \{t_k\}_{k \in \mathbb{N}_0} \\
x(0) &= x_0
\end{align*}\]  \hspace{1cm} (1)

where \(x(t^-) = \lim_{s \uparrow t} x(s)\).

- A continuous part
- A discrete part
- A set of impulse instants \(\{t_k\}_{k \in \mathbb{N}_0}, \quad t_0 = 0\).

Jumping rule

- State-dependent jumping instants, e.g. when \(x\) enters some sets (internal)
- Time-dependent jumping instants (external)
• Stability depends on the matrices of the system but also on the set of impulse instants!
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- How can we characterize stability in an efficient/accurate/tractable way?
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• How can we characterize stability in an efficient/accurate/tractable way?
• How can we derive tractable conditions for control design?
Stability of impulsive systems
Dwell-times

**Definition**
The dwell-time $T_k$ is defined as $T_k = t_{k+1} - t_k$, i.e. $t_{k+1} = t_k + T_k$. 
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Average dwell-time$^1$

- The number of impulses in any time interval
- Asymptotic notion

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Average dwell-time
- The number of impulses in any time interval
- Asymptotic notion

Minimum/maximum/range dwell-time
- Minimum dwell-time: $T_k \geq \bar{T}$ for some $\bar{T} > 0$, $k \in \mathbb{N}_0$
- Maximum dwell-time: $T_k \leq \bar{T}$ for some $\bar{T} > \varepsilon > 0$, $k \in \mathbb{N}_0$
- Minimum dwell-time: $T_k \in [T_{min}, T_{max}]$, for some $0 < T_{min} \leq T_{max} < \infty$, $k \in \mathbb{N}_0$
- Non-asymptotic notion

---

Theorem (1)

Assume that there exist $P \in \mathbb{S}_>^n$ and scalars $c, d \in \mathbb{R}$, $d \neq 0$, such that

$$
A^T P + PA + cP < 0
$$

$$
J^T PJ - e^{-d} P < 0.
$$

Then, the system is stable provided that the number of impulses $N(t, s)$ over the interval $(s, t]$ satisfies

$$
-dN(t, s) - (c - \lambda)(t - s) \leq \mu, \text{ for all } t \geq s
$$

for some arbitrary constants $\lambda, \mu > 0$.

---

Theorem (1)

Assume that there exist $P \in S^n_{>0}$ and scalars $c > 0, d < 0$, such that

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$$

(3)

Then, the system is stable provided that the number of impulses $N(t, s)$ over the interval $(s, t]$ satisfies

$$
N(t, s) \leq \frac{t - s}{\tau^*} + N_0, \text{ for all } t \geq s.
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Assume that there exist $P \in \mathbb{S}^n_{>0}$ and a scalar $\bar{T} > 0$ such that the conditions

$$
A^T P + PA < 0 \quad \text{and} \quad J^T e^{A T} \bar{T} P e^{A^T \bar{T}} J - P < 0
$$

(5)

hold.

Then, the system is stable provided that $T_k \geq \bar{T}$; i.e. $t_{k+1} \geq t_k + \bar{T}$, $k \in \mathbb{N}_0$.

- A must be Hurwitz
- Stable continuous-time dynamics, potentially unstable discrete-time dynamics
- If we let $\bar{T} \to 0$, then we obtain a condition for arbitrary impulse times (but we must deal with Zeno behavior)
- Easy to check

---

Theorem (1)

Assume that there exist $P \in S^n_{>0}$ and a scalar $\bar{T} > 0$ such that the conditions

$$
A^T P + PA > 0 \\
J^T e^{A^T \bar{T}} Pe^{A\bar{T}} J - P < 0
$$

(6)

hold.

Then, the system is stable provided that $0 < \varepsilon < T_k \leq \bar{T}$; i.e. $t_{k+1} \leq t_k + \bar{T}$, $k \in \mathbb{N}_0$.

- $A$ must be anti-Hurwitz
- Anti-stable continuous-time dynamics, stable discrete-time dynamics
- Easy to check
Discretization

• We consider here the discrete-time system

\[ x(t_{k+1}^-) = e^{AT_k}Jx(t_k^-), \quad k \in \mathbb{N}_0 \]  

(7)

where \( t_0 = 0 \) and \( T_k \in [T_{\text{min}}, T_{\text{max}}] \).
Discretization

- We consider here the discrete-time system

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Theorem (1)

Assume that there exist \( P \in \mathbb{S}^n_{>0} \) such that the condition

\[
J^T e^{AT} \theta P e^{A\theta} J - P < 0
\]

holds for all \( \theta \in [T_{\text{min}}, T_{\text{max}}] \).

Then, the system is stable provided that \( T_k \in [T_{\text{min}}, T_{\text{max}}], k \in \mathbb{N}_0 \).

- Robust feasibility problem (due to parametric dependence)
- Not easy to check since non-convex in \( \theta \) . . .
Robustness

- Robust LMIs are difficult to check

\[ J^T e^{A^T \theta} P e^{A \theta} J - P < 0, \quad \theta \in [T_{min}, T_{max}] \]
Robustness

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\[ J^T e^{A^T \theta} P e^{A \theta} J - P < 0, \quad \theta \in [T_{min}, T_{max}] \]

- Difficult to extend to uncertain matrices \( A \)

\[ J^T e^{(A+\Delta)^T \bar{T}} P e^{(A+\Delta) \bar{T}} J - P < 0 \]
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\[ J^T e^{(A+\Delta)^T \bar{T}} P e^{(A+\Delta)\bar{T}} J - P < 0 \]

- Not directly applicable to systems with time-varying \( A \)

\[ J^T \Phi(\bar{T})^T P \Phi(\bar{T}) J - P < 0 \]
Difficulties

Robustness

• Robust LMIs are difficult to check

\[ J^T e^{A^T \theta} P e^{A\theta} J - P < 0, \quad \theta \in [T_{min}, T_{max}] \]

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Control Design

• Not convex

\[ J^T e^{(A+BK)^T \bar{T}} P e^{(A+BK)\bar{T}} J - P < 0 \]
Convex conditions for periodic impulses

Theorem
Let us consider an impulsive system $(A, J)$ with periodic impulses, i.e. $T_k = \bar{T}, k \in \mathbb{N}$. Then, the following statements are equivalent:

(a) The impulsive system with $\bar{T}$-periodic impulses is asymptotically stable.
Theorem
Let us consider an impulsive system \((A, J)\) with periodic impulses, i.e. \(T_k = \bar{T}, k \in \mathbb{N}\). Then, the following statements are equivalent:

(a) The impulsive system with \(\bar{T}\)-periodic impulses is asymptotically stable.

(b) There exists a matrix \(P \in \mathbb{S}_n^{>0}\) such that the LMI

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JT e^{A^T \bar{T}} Pe^{A\bar{T}} J - P < 0
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(a) The impulsive system with \(\bar{T}\)-periodic impulses is asymptotically stable.

(b) There exists a matrix \(P \in \mathbb{S}^n \succ 0\) such that the LMI

\[
J^T e^{A^T \bar{T}} P e^{A \bar{T}} J - P \prec 0
\]

holds.

(c) There exist a differentiable matrix function \(R : [0, \bar{T}] \mapsto \mathbb{S}^n\), \(R(0) \succ 0\), and a scalar \(\varepsilon > 0\) such that the LMIs

\[
A^T R(\tau) + R(\tau) A + \dot{R}(\tau) \leq 0 \quad \text{and} \quad J^T R(0) J - R(\bar{T}) + \varepsilon I \leq 0
\]

hold for all \(\tau \in [0, \bar{T}]\).
Theorem

Let us consider an impulsive system $(A, J)$ with periodic impulses, i.e. $T_k = \bar{T}$, $k \in \mathbb{N}$. Then, the following statements are equivalent:

(a) The impulsive system with $\bar{T}$-periodic impulses is asymptotically stable.

(b) There exists a matrix $P \in \mathbb{S}_n^+$ such that the LMI

$$J^T e^{A^T \bar{T}} P e^{A\bar{T}} J - P \prec 0$$

holds.

(c) There exist a differentiable matrix function $R : [0, \bar{T}] \mapsto \mathbb{S}_n$, $R(0) > 0$, and a scalar $\varepsilon > 0$ such that the LMIs

$$A^T R(\tau) + R(\tau) A + \dot{R}(\tau) \preceq 0 \quad \text{and} \quad J^T R(0) J - R(\bar{T}) + \varepsilon I \preceq 0$$

hold for all $\tau \in [0, \bar{T}]$.

(d) There exist a differentiable matrix function $S : [0, \bar{T}] \mapsto \mathbb{S}_n$, $S(\bar{T}) > 0$, and a scalar $\varepsilon > 0$ such that the LMIs

$$A^T S(\tau) + S(\tau) A - \dot{S}(\tau) \preceq 0 \quad \text{and} \quad J^T S(\bar{T}) J - S(0) + \varepsilon I \preceq 0$$

hold for all $\tau \in [0, \bar{T}]$. 

Convex conditions for range dwell-time

Theorem
Let us consider an impulsive system \((A, J)\). Then, the following statements are equivalent:

(a) There exists a matrix \(P \in \mathbb{S}_n^+\) such that the LMI

\[
J^T e^{A^T \theta} P e^{A \theta} J - P < 0
\]

holds for all \(\theta \in [T_{\text{min}}, T_{\text{max}}]\).

Moreover, when one of the above statements holds, then the aperiodic impulsive system with ranged dwell-time \(T_k \in [T_{\text{min}}, T_{\text{max}}]\) is asymptotically stable.
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(b) There exist a differentiable matrix function \(R : [0, T_{\text{max}}] \mapsto \mathbb{S}^n\), \(R(0) > 0\), and a scalar \(\epsilon > 0\) such that the LMIs

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A^T R(\tau) + R(\tau) A + \dot{R}(\tau) \preceq 0
\]

and

\[
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\]

hold for all \(\tau \in [0, T_{\text{max}}]\) and all \(\theta \in [T_{\text{min}}, T_{\text{max}}]\).

Moreover, when one of the above statements holds, then the aperiodic impulsive system with ranged dwell-time \(T_k \in [T_{\text{min}}, T_{\text{max}}]\) is asymptotically stable.
Theorem (Minimum Dwell-Time)

Let us consider an impulsive system \((A, J)\). Then, the following statements are equivalent:

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A^T P + PA < 0 \quad \text{and} \quad J^T e^{A^T \bar{T}} Pe^{A \bar{T}} J - P < 0
\]

hold.

Moreover, when one of the above statements holds, the impulsive system is asymptotically stable under minimum dwell-time \(\bar{T}\), i.e. for any sequence \(\{t_k\}_{k \in \mathbb{N}}\) such that \(T_k \geq \bar{T}\).
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hold.

(b) There exist a differentiable matrix function \(R : [0, \bar{T}] \to \mathbb{S}^n, R(0) > 0\), and a scalar \(\varepsilon > 0\) such that the LMIs

\[
A^T R(0) + R(0) A < 0
\]

\[
A^T R(\tau) + R(\tau) A + \dot{R}(\tau) \leq 0 \quad \text{and} \quad J^T R(0) J - R(\bar{T}) + \varepsilon I \leq 0
\]

hold for all \(\tau \in [0, \bar{T}]\).

Moreover, when one of the above statements holds, the impulsive system is asymptotically stable under minimum dwell-time \(\bar{T}\), i.e. for any sequence \(\{t_k\}_{k \in \mathbb{N}}\) such that \(T_k \geq \bar{T}\).
Pros and cons

Benefits

- Convex in the matrices of the system → robustness analysis possible
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Drawbacks
- Infinite-dimensional LMI problems
Pros and cons

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- Convex in the matrices of the system → robustness analysis possible
- Convex in the matrices of the system → control design possible
- Applicable to systems with time-varying matrices

Drawbacks

- Infinite-dimensional LMI problems
- Needs relaxation (piecewise linear approximation or SOS)
Example 1 - Range dwell-time

Let us consider the system \(^1\)

\[
A = \begin{bmatrix} -1 & 0.1 \\ 0 & 1.2 \end{bmatrix}, \quad J = \begin{bmatrix} 1.2 & 0 \\ 0 & 0.5 \end{bmatrix}.
\] (13)
Example 1 - Range dwell-time

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<th>(T_{min})</th>
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<td></td>
<td>6</td>
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</tr>
<tr>
<td>Periodic case</td>
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<td>0.1824</td>
</tr>
</tbody>
</table>

- Finds the theoretical bounds
- Also holds in the aperiodic case

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\(^1\) C. Briat et al. *A looped-functional approach for robust stability analysis of linear impulsive systems*, *Systems & Control Letters*, 2012
Example 2 - Minimum dwell-time

Let us consider the system

\[ A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad J = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}. \quad (14) \]
Example 2 - Minimum dwell-time

Let us consider the system\(^1\)

\[
A = \begin{bmatrix}
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1 & -2
\end{bmatrix}, \quad J = \begin{bmatrix}
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\end{bmatrix}.
\]

(14)

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\cdot Non-conservative dwell-time result

Robustification

Let us consider now the system

\[
\begin{align*}
\dot{x}(t) &= Ax(t), \quad t \notin \{t_k\}_{k \in \mathbb{N}_0} \\
x(t) &= Jx(t^-), \quad t \in \{t_k\}_{k \in \mathbb{N}_0} \\
x(0) &= x_0
\end{align*}
\]  

where

\[
A \in \mathcal{A} := \text{co} \{A_1, \ldots, A_N\}, \quad J \in \mathcal{J} := \text{co} \{J_1, \ldots, J_N\}
\]
Robustification

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where

\[A \in \mathcal{A} := \text{co} \{A_1, \ldots, A_N\}, \ J \in \mathcal{J} := \text{co} \{J_1, \ldots, J_N\}\]

- Define the set

\[
\Phi_{\bar{T}} := \{\Phi(t) : \Phi(s) \text{ solves } (16), \ \lambda(s) \in \Lambda_N, \ s \in [0, \bar{T}]\}.
\]

\[
\frac{d\Phi(s)}{ds} = \left(\sum_{i=1}^{N} \lambda_i(s)A_i\right)\Phi(s), \ \Phi(0) = I.
\]
Robustification

- Let us consider now the system

\[ \begin{align*}
\dot{x}(t) &= Ax(t), \ t \notin \{t_k\}_{k \in \mathbb{N}_0} \\
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\end{align*} \tag{15} \]

where

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- Define the set

\[ \Phi_{\bar{T}} := \{ \Phi(\bar{T}) : \Phi(s) \text{ solves (16), } \lambda(s) \in \Lambda_N, \ s \in [0, \bar{T}] \} . \]

\[ \frac{d\Phi(s)}{ds} = \left( \sum_{i=1}^{N} \lambda_i(s) A_i \right) \Phi(s), \ \Phi(0) = I. \tag{16} \]

- We can now consider the uncertain discrete-time system

\[ x((k + 1)\bar{T}) = \Psi Jx(k\bar{T}), \ k \in \mathbb{N}_0 \tag{17} \]

where \( \Psi \in \Phi_{\bar{T}}. \)
Robustification

Theorem

Let us consider an uncertain (time-varying) impulsive system \((A, J)\), \(A \in \mathcal{A}, J \in \mathcal{J}\), with \(\bar{T}\)-periodic impulses. Then, the following statements are equivalent:

(a) The uncertain (time-varying) impulsive system with \(\bar{T}\)-periodic impulses is quadratically stable.
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(a) The uncertain (time-varying) impulsive system with \(\bar{T}\)-periodic impulses is quadratically stable

(b) There exists a matrix \(P \in \mathbb{S}_{>0}^n\) such that the LMI

\[
J^T \Psi^T P \Psi J - P \prec 0
\]

holds for all \((\Psi, J) \in \Phi_{\bar{T}} \times \mathcal{J}\).
Theorem

Let us consider an uncertain (time-varying) impulsive system \((A, J)\), \(A \in \mathcal{A}, J \in \mathcal{J}\), with \(\bar{T}\)-periodic impulses. Then, the following statements are equivalent:

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(c) There exist a differentiable matrix function \(R : [0, \bar{T}] \mapsto \mathbb{S}_n\), \(R(0) > 0\), and a scalar \(\varepsilon > 0\) such that the LMIs

\[
A_i^T R(\tau) + R(\tau) A_i + \dot{R}(\tau) \preceq 0, \quad \text{and} \quad J_i^T R(0) J_i - R(\bar{T}) + \varepsilon I \preceq 0
\]

hold for all \(\tau \in [0, \bar{T}]\) and all \(i = 1, \ldots, N\).
Stabilization of impulsive systems
Stabilization problem

System

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_c u_c(t), \ t \neq t_k \\
x(t) &= Jx(t^-) + B_d u_d(t), \ t = t_k
\end{align*}
\]

where \( u_c \in \mathbb{R}^{m_c} \) and \( u_d \in \mathbb{R}^{m_d} \) are the control inputs.
Stabilization problem

System

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_c u_c(t), \quad t \neq t_k \\
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\end{align*}
\]  

(18)

where \( u_c \in \mathbb{R}^{m_c} \) and \( u_d \in \mathbb{R}^{m_d} \) are the control inputs.

Control law

We consider the following class of control-laws:

\[
\begin{align*}
    u_c(t_k + \tau) &= K_c(\tau)x(t_k + \tau), \quad \tau \in [0, T_k), \\
    u_d(t_k) &= K_d x(t_k^-)
\end{align*}
\]  

(19)
System

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bc u_c(t), \ t \neq t_k \\
x(t) &= Jx(t^-) + Bd u_d(t), \ t = t_k
\end{align*}
\]  

(18)

where \( u_c \in \mathbb{R}^{mc} \) and \( u_d \in \mathbb{R}^{md} \) are the control inputs.

Control law

We consider the following class of control-laws:

\[
\begin{align*}
u_c(t_k + \tau) &= K_c(\tau)x(t_k + \tau), \ \tau \in [0, T_k), \\
u_d(t_k) &= K_d x(t_k^-)
\end{align*}
\]  

(19)

Minimum dwell-time case

\[
K_c(\tau) = \begin{cases} 
\tilde{K}_c(\tau) & \text{if } \tau \in [0, \bar{T}) \\
\tilde{K}_c(\bar{T}) & \text{if } \tau \in [\bar{T}, T_k) 
\end{cases}
\]  

(20)

where \( T_k \geq \bar{T}, k \in \mathbb{N} \) and \( \tilde{K}_c(\tau) \) is some matrix function to be determined.
Theorem (Minimum dwell-time)

Assume that there exist a differentiable matrix function \( S : [0, \bar{T}] \mapsto \mathbb{S}^n \), \( S(\bar{T}) \succ 0 \), a matrix function \( U_c : [0, \bar{T}] \mapsto \mathbb{R}^{m_c \times n} \), a matrix \( U_d \in \mathbb{R}^{m_d \times n} \) and a scalar \( \varepsilon > 0 \) such that the LMIs

\[
\text{Sym}\left[ AS(\bar{T}) + B_c U_c(\bar{T}) \right] \prec 0, \quad (21)
\]

\[
\text{Sym}\left[ AS(\tau) + B_c U_c(\tau) \right] + \dot{S}(\tau) \preceq 0 \quad (22)
\]

and

\[
\begin{bmatrix}
-S(0) + \varepsilon I & JS(\bar{T}) + B_d U_d \\
* & -S(\bar{T})
\end{bmatrix} \preceq 0 \quad (23)
\]

hold for all \( \tau \in [0, \bar{T}] \). Then, the closed-loop system is asymptotically stable with minimum dwell-time \( \bar{T} \) and suitable controller gains are retrieved using

\[
\tilde{K}_c(\tau) = U_c(\tau)S(\tau)^{-1} \quad \text{and} \quad K_d = U_d S(\bar{T})^{-1}. \quad (24)
\]
Range dwell-time result

Theorem (Range dwell-time)

Assume that there exist a differentiable matrix function $S : [0, \bar{T}] \mapsto \mathbb{S}^n$, $S(0) \succ 0$, a matrix function $U_c : [0, \bar{T}] \mapsto \mathbb{R}^{m_c \times n}$, a matrix $U_d \in \mathbb{R}^{m_d \times n}$ and a scalar $\varepsilon > 0$ such that the LMIs

\[
\text{Sym}[AS(\tau) + B_c U_c(\tau)] + \dot{S}(\tau) \preceq 0
\]

and

\[
\begin{bmatrix}
-S(\theta) + \varepsilon I & JS(0) + B_d U_d \\
\ast & -S(0)
\end{bmatrix} \preceq 0
\]

hold for all $\tau \in [0, T_{\text{max}}]$ and all $\theta \in [T_{\text{min}}, T_{\text{max}}]$. Then, the closed-loop system is asymptotically stable with range dwell-time $[T_{\text{min}}, T_{\text{max}}]$ and suitable controller gains are retrieved using

\[
\tilde{K}_c(\tau) = U_c(\tau)S(\tau)^{-1} \quad \text{and} \quad K_d = U_d S(0)^{-1}.
\]
Let us consider the system with matrices

\[
A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}
\]  

(28)

- We want to compute \( \tilde{K}_c(\tau) \) such that the minimum dwell-time is, at most, \( \bar{T} = 0.1 \).
Let us consider the system with matrices

\[
A = \begin{bmatrix}
  1 & 0 \\
  1 & 2 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
  1 \\
  0 \\
\end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix}
  1 & 1 \\
  1 & 3 \\
\end{bmatrix}
\]  

(28)

- We want to compute \( \tilde{K}_c(\tau) \) such that the minimum dwell-time is, at most, \( \bar{T} = 0.1 \).

- We obtain

\[
\tilde{K}_c(\tau) = \frac{1}{d(\tau)} \begin{bmatrix}
  1.4750481 + 3.2714889\tau - 41.011914\tau^2 \\
  3.9063911 - 1.6733059\tau - 37.472443\tau^2 \\
\end{bmatrix}^T
\]

where \( d(\tau) = -0.19767438 + 0.78454217\tau + 7.6562219\tau^2 \).
Let us consider the system with matrices

\[ A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } J = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \quad (28) \]

- We want to compute \( \tilde{K}_c(\tau) \) such that the minimum dwell-time is, at most, \( \bar{T} = 0.1 \).
Sampled-data systems
Sampled-data systems

System

Let us consider now the continuous-time system

\[ \dot{x}(t) = Ax(t) + Bu(t) \]  \hspace{1cm} (29)

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are the state of the system and the control input, respectively.
System
Let us consider now the continuous-time system

\[ \dot{x}(t) = Ax(t) + Bu(t) \quad (29) \]

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) are the state of the system and the control input, respectively.

Controller
The control input is assumed to be computed from a sampled-data state-feedback control law given by

\[ u(t) = K_1 x(t_k) + K_2 u(t_{k-1}), \quad t \in [t_k, t_{k+1}) \quad (30) \]

where \( K_1 \in \mathbb{R}^{m \times n} \) and \( K_2 \in \mathbb{R}^{m \times m} \) are the control gains to be determined.
Sampled-data systems

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Objectives
Find a control law of the form (30) such that the closed-loop system is robustly stable for all sampling-periods in the range \([T_{min}, T_{max}]\).
Sampled-data systems as impulsive systems

- Any sampled-data system can be equivalently reformulated as an impulsive system:

\[
\begin{align*}
\dot{x}(t) & = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} x(t), \quad t \neq t_k \\
\dot{z}(t) & = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} z(t), \\
\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} & = \begin{bmatrix} I & 0 \\ K_1 & K_2 \end{bmatrix} \begin{bmatrix} x(t^-) \\ z(t^-) \end{bmatrix}, \quad t = t_k
\end{align*}
\]

(31)

where \( z(t) = u(t_k), \ t \in [t_k, t_{k+1}) \).

- Let \( \bar{J} = J_0 + B_0 K \) where

\[
J_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}.
\]

(32)
Theorem (Aperiodic sampled-data systems)

The following statements are equivalent:

(a) There exists a control law of the form (30) that quadratically stabilizes the system (29) for any aperiodic sampling instant sequence \( \{t_k\} \) such that \( T_k \in [T_{\text{min}}, T_{\text{max}}] \).
Theorem (Aperiodic sampled-data systems)
The following statements are equivalent:

(a) There exists a control law of the form (30) that quadratically stabilizes the system (29) for any aperiodic sampling instant sequence \( \{t_k\} \) such that \( T_k \in [T_{min}, T_{max}] \).

(b) There exist a differentiable matrix function \( R : [0, T_{max}] \mapsto \mathbb{S}^{n+m}, S(0) \succ 0 \), a matrix \( Y \in \mathbb{R}^{m \times (n+m)} \) and a scalar \( \varepsilon > 0 \) such that the conditions

\[
\bar{A}(\tau)S(\tau) + S(\tau)\bar{A}(\tau)^T + \dot{S}(\tau) \preceq 0
\]

(33)

and

\[
\begin{bmatrix}
-S(\theta) + \varepsilon I & J_0 S(0) + B_0 Y \\
* & -S(0)
\end{bmatrix} \preceq 0
\]

(34)

hold for all \( \tau \in [0, T_{max}] \) and all \( \theta \in [T_{min}, T_{max}] \).

Moreover, when this statement holds, a suitable stabilizing control gain can be obtained using the expression \( K = Y S(0)^{-1} \).
Example 1

Let us consider the sampled-data system (29) with matrices

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \]
Example 1

Let us consider the sampled-data system (29) with matrices

\[
A = \begin{bmatrix}
0 & 1 \\
0 & -0.1
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
0 \\
0.1
\end{bmatrix}.
\] (35)

- Fixed control law: \( K_1 = \begin{bmatrix}
-3.75 \\
-11.5
\end{bmatrix} \) and \( K_2 = 0. \)

<table>
<thead>
<tr>
<th>( d_R )</th>
<th>System (35) ( T_{max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed result</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>6</td>
</tr>
<tr>
<td>(Fridman et al., 2004)</td>
<td>–</td>
</tr>
<tr>
<td>(Naghshtabrizi et al., 2008)</td>
<td>–</td>
</tr>
<tr>
<td>(Fridman, 2010)</td>
<td>–</td>
</tr>
<tr>
<td>(Liu et al., 2010)</td>
<td>–</td>
</tr>
<tr>
<td>(Seuret, 2012)</td>
<td>–</td>
</tr>
<tr>
<td>(Seuret and Peet, 2013)</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>
Example 1

Let us consider the sampled-data system (29) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}. \quad (35)$$

- Designed control law for some given $[T_{min}, T_{max}]$.

<table>
<thead>
<tr>
<th>$T_{min}$</th>
<th>$T_{max}$</th>
<th>$K_1$</th>
<th>$K_2$</th>
<th>$d_R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>10</td>
<td>-0.1145</td>
<td>-0.8088</td>
<td>-0.0024</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>-0.0202</td>
<td>-0.1560</td>
<td>-0.0030</td>
</tr>
<tr>
<td>0.001</td>
<td>10</td>
<td>-0.0310</td>
<td>-0.3222</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>50</td>
<td>-0.0259</td>
<td>-0.2726</td>
<td>0</td>
</tr>
</tbody>
</table>
Example 2

- Let us consider the following sampled-data system (29) with matrices

\[ A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  

(36)

- Let \( K_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \) and \( K_2 = 0. \)

<table>
<thead>
<tr>
<th>( d_R )</th>
<th>( T_{min} )</th>
<th>( T_{max} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Proposed result</td>
<td>4 6</td>
<td>0.4 0.4</td>
</tr>
<tr>
<td>(Seuret, 2012)</td>
<td>–</td>
<td>0.400</td>
</tr>
<tr>
<td>(Seuret and Peet, 2013)</td>
<td>3 5</td>
<td>0.4 0.4</td>
</tr>
</tbody>
</table>
Let us consider the uncertain sampled-data system (29) with matrices

\[
A \in \mathcal{A} = \text{co} \left\{ \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \delta \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \right\} \text{ and } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

(37)

where \( \delta \) is a positive parameter.

<table>
<thead>
<tr>
<th>( \delta )</th>
<th>( T_{min} )</th>
<th>( T_{max} )</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
<th>( d_R )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
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<td>10</td>
<td>-0.0757</td>
<td>-0.0006</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
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<td>20</td>
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<td>10</td>
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</tr>
<tr>
<td>20</td>
<td>0.001</td>
<td>20</td>
<td>-0.0339</td>
<td>-0.0019</td>
<td>2</td>
</tr>
</tbody>
</table>
Concluding remarks
Concluding statements

- Robust stability under minimum, maximum and range dwell-time
- Robust stabilization possible
- Can be extended to homogeneous Lyapunov functions easily

Possible extensions

- Switched systems, time-dependent hybrid systems
- Dynamic output feedback?
- Nonlinear systems
Thank you for your Attention